Notes on

Sets and Plurals

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§1 Natural language users employ plural terms (“the apples”, “the squad”, “her followers” etc.), this nobody denies. So, the question of their semantics arises, and here the controversies start. These notes here support the view that plurals can be *semantically and logically* modelled in some ways by sets or by simple ordinary First Order Logic (FOL) quantification (i.e. without even appeal to sets). There should be wide-spread agreement that collections exist. They are, at least, part of our ‘manifest image’ of the world. Their semantic and logical role, however, should be separated from their metaphysics, which may be dealt with in physics or cognitive science. The logical aspects of collections – so the thesis here – are captured by (some) set theory. Sets may be seen as most natural reference units for plural expressions. “The chairs in the room” seems to refer to the set of chairs in the room.

The use of sets as reference for plural terms and the corresponding denial of plural quantification in addition to singular quantification (explaining the *phenomena* as a quantification either with respect to sets or with a set as restricted domain) has been questioned, though.[1]

Some main arguments for the main thesis and replies to opponents of a set-based account are presented. The emphasis lies on the logic of plurals, denial that a further Plural Quantificational Logic (PQL) is needed, and the avoidance of further metaphysics.

[1] Prominently so in: Oliver & Smiley, *Plural Logic*; cf. also: Florio & Linnebo, “Critical Pural Logic”; critical: Ben-Yami, “Plural Quantification Logic: A Critical Appraisal”.

§2 The first important distinction to have in view is the difference between a collective use of a plural term and a distributive use.

(1) The churchgoers brought their prayer-books.

says of each member of the collection that this member brought his/her prayer-book. (1) is false if this (bounded) universal quantification fails.

(2) The apples were too heavy for the paper bag.

says of the collection of apples *as a whole* that it was too heavy.

Thus, (1) being distributive, (2) being collective usage.

In English "every" links preferably to distributive usage, "all" may be used in collective cases, if such occur at all with a quantifier. Quantifier presence rather invites a distributive reading, e.g.

(3) All apples had to be removed.

In (1) the set may be seen as domain, in (2) as object of reference.

§3 Collective usage immediately puts the question whether sets can serve as referents in space-time contexts. Aren't they abstract entities? – Supposedly, sets aren't abstract, but non-existent, only a convenient fiction which can be partially mapped to reality. The 'set of apples' is mapped to a real (in this case physical) collection of apples. Or, said otherwise: we *talk* of a real collection as if it was a set. Properties of sets (like extensionality) are useful to work with. So, we work with the concept of 'set' when talking about real collections. Set theory contains aspects of sets (being a subset, being orderable, having a powerset etc.) which can be usefully applied to real collections. (Only if set theory could be reduced to some mereology sets would be 1st class references.) Sets with existing individuals as elements are anchored to real collections of these. Pure sets which model lower real or rational numbers, lower finite cardinals and ordinals can be partially mapped to numerical relations of existing individuals and arithmetical operations with them; similar mapping applies to lower finite instances of geometric concepts, and concepts of other branches of set-based mathematics. Purely pure sets without this mapping are purely fictional as scaffolding in set theory, they serve to increase the *coherence of the narrative* of set

theory (best in the Cumulative Hierarchy *V*). Fictionalism about sets comes with the idea of a partial homomorphism between set theory (including operations with sets) and reality (including logical operations concerning objects) – a model, of course, in a sense before the explication of 'model' in set theory.[2]

Thus, the issue here is not the metaphysics of sets in general, but their useability in the context of plural expressions. Real collections back up our understanding of sets. Collections can be separated or made out in situations. Collections should not be denied as entities. There might be scientific theories about the discrimination, apprehension and constitution of real collections. None of this matters much or at all for their semantic and logical role, *inter alia* in plurals. These *semantic and logical properties are properly externalized into set theory*, so the thesis here.

The insistence on a set-based semantics for plurals does neither involve a denial of the existence of real collections nor an attempt to reduce our understanding of them to an understanding of sets. The dependency in concept acquisition might be seen in both directions, typically the *explicit* (theoretical) concept of 'set' follows an apprehension of real collections.

§4 Distributive usage seems to pose no new semantic issues above what can be said about Generalized Quantifiers and bounded quantification. Statements like (1) seem to be translatable into universally quantified conditionals where only individual variables occur. Statements like (2) quantify over the collection, but given either the relation of membership [$\in$] or the relation of being a part [$\prec$] – in some suitable mereology – all quantifiers run over 1st level individuals. There seems to be no need to treat the collection/set as being a higher order entity. So, what should be complained about this supposedly simple account of plurals?

§5 In any case, the debate should not be just metaphysical, but centre on usage and logical properties of plurals in natural language. A theory of plurals should (i) be descriptive adequate with respect to usage, and (ii) be explanatory with respect to the logical properties of plural expressions. If two theories both seem to fulfil these two conditions, then the typical further

[2] Cf. Bremer, "Fictionalism in Mathematics".

criteria of theory choice should be applied: (iii) the theory should be ontologically simple and coherent/consilient with neighbouring theories (say, in this case, theories of natural language quantification and ontologies of collections). All new constructs and ontological posits going beyond established theories carry a burden of proof. In this respect set talk has been established as useful and ingrained in all mathematical sub-areas, formal linguistics, computability theory, and meta-logic. FOL (by *Hilbert's Thesis* that *any* cogent argument in mathematics can be rendered in FOL, and *Turing's Equivalence Lemma* that FOL and Turing-Machines *solve the same problems*) is *the* logic deviations from which (typically restriction of application of full FOL in some contexts) carry the burden of proof.

§6 The formal language applicable to plurals is FOL with variables and constants divided into sorts for convenience. The language talks about ordinary objects *and* sets, and their properties. In this formal language [X] is not a 2nd order variable, but a variable taking sets as values. One could have used simple FOL with a predicate [Set( )] and variables [x], [y] ... only (i.e. [$\exists$x(Set(x) $\wedge$ ...] and [$\forall$x(Set(x) $\supset$ ...]).

[$\in$] appears only with a capitalized singular term on the right-hand side: [a$\in$b] is not well-formed. The language allows also set abstracts, like $\{x \mid \text{Apple}(x)\}$.

[=] for capitalized singular terms is defined by the *Axiom of Extensionality*, now in the form:

$$\vdash(\forall X, Y)(\forall z, Z)((z\in X \equiv z\in Y) \wedge (Z\in X \equiv Z\in Y) \supset X = Y)$$

as sets can contain both urelements as well as other sets. (The convenience of two sorts of singular terms and variables when paying attention to the difference between objects and sets may turn to a nuisance in stating and proving set theoretical principles.)

Mixed [=]-statements like [a = R] are not well-formed.

Predicates can be applied to set expressions, and sets can be elements of other sets, thus [X$\in$Y] and [I(X)] are well-formed.

Paradoxes are avoided by having the ZFC-axioms/axiom-schemata for sets. The domain of the language, therefore, is the Cumulative Hierarchy *V* with non-set individuals thrown in as

urelements at the lowest level V_1 (i.e. the language is a two-sorted ZFCU). (For a set theoretical fictionalist the lowest level (domain of an ordinary FOL-type language) exists, and the higher ranks V_r constitute the fiction of the Cumulative Hierarchy.)

Let us call this system mnemotechnically "FTZU" (**F**irst Order **T**wo Sorted **Z**FC with **U**relements).

§7 Given an analysis of plurals in terms of sets the truth-condition for (1) comes to

(1') $\forall x(C(x) \supset (\exists y)(P(y) \wedge B(x,y))$

sets do not appear here (in the object language). Sets appear in the meta-language: $||C|| = C \subseteq V_1$. The semantic value of the predicate [C( )] is a subset of the domain of urelements (not set-like individuals). Sets can be members of sets (including semantic values of corresponding predicates) in the hierarchy V. A set at rank V_r is a subset of V_{r-1} (as its domain, which, of course, collects all the individuals present in the ranks V_{r*} with $r^* < r-1$).

§8 Sets in FTZU (like in ZFC and ZFCU), for all set theorists know, behave consistently. As philosophical tool FTZU meta-theory needs to be supplemented by a *story* of the special status of the unique object V: an exposition of the idea of the Cumulative Hierarchy mirrored in the ZFC-axioms, and the special status of V itself, and "V" itself – which can be done (starting from a fictionalist reading of basic set theoretical literature by the founding fathers of set theory: Cantor, Zermelo, Fraenkel, von Neumann *et altera*).[3] And the *narrative* around V (being uncollectible but the target of reference by "V", etc.) does not threaten the existence of a domain of urelements, even understood as in space and/or time.

If one finds, nonetheless, any contradictions ensuing by using FTZU there will always be the option to move to a paraconsistent logic (say, an Adaptive Logic version of FTZU) or even a

[3] Cf. Bremer, "Universality and the Universe in ZFC". This story can also be justifiably preferred as more coherent than other stories, like those of other set and class theories: NBG, MK, KP, NF ... cf. also: Boolos, "The Iterative Conception of Sets", and: Maddy, *Defending the Axioms*, and: Potter, *Set Theory and Its Philosophy*.

paraconsistent logic accompanying a paraconsistent set theory (typically containing a version of *Naive Comprehension*).[4]

Issues of consistency (of set theory) should be kept apart from the issue of plural terms.

§9 The truth-condition for (2) comes to

(2') $(\exists X)(\forall y)(y \in X \supset A(y) \wedge (\exists z)(P(z) \wedge T(X,z))$

Sets do appear in the object language, domains/ranks of which include sets as individuals. In (2') a set enters in a relation *Too-heavy-for* which is physical. This need not mean that sets as entities enter in such relations, but – as indicated above – that we engage in this manner of speaking. The justification being:

(i) We perfectly understand when a statement like (2) is true, and the truth-condition (2') is acceptable enough to know when (2) is true.

(ii) Ontologically this type of set-talk can be accepted as real collections provide a partial model for set-talk. Not all properties of real collections are relevant for truth-conditions of plural usage. The relevant properties are captured by set-talk. No extended theory (or metaphysics) of either real collections or sets and classes is required for an account of plural usage. Sets are 'abstract' in the sense that the concept of 'set' contains those aspects *abstracted from* real collections which are relevant for their logical behaviour or their role in logic. Set-talk, thus, does not *refer* to abstract entities, but is an *abstract way* to talk about real collections, some of which are the referents of plural expressions.

Thus, a semantics of plurals might be kept apart from wider metaphysical theories – if one wants to engage in those at all.

[4] Prominently Routley's combination of a 'Ultralogic' with a 'Dialectical Set Theory' cf. Routley, *Ultralogic as Universal?* and Brady, *Universal Logic*. Cf. also: Priest, *In Contradiction*, chapter 18; Bremer, *An Introduction to Paraconsistent Logics*, chapter 11.

§10 Focussing on sets has further advantages:

(i) There is an empty set ∅, whereas an empty real collection seems barely comprehensible.

(ii) There are singletons for all individuals, whereas a real collection of a single object seems to be a degenerate case difficult to understand as differing from the object in the first place.

(iii) Sets can be quantified over existentially and universally, whereas universal quantification over plurals seems missing in natural languages.

The principles of a *logic* of plurals (backed up or not by an ontology of real collections) might be reduced to set theoretical axioms and principles. Even if the proponent of a set-based account of plurals neither denies the use of plural terms nor that real collections ultimately ground set talk, she may well deny that plural talk needs any new logical principles beyond FTZU.

§11 Someone might demand that a proper semantics should embed truth-conditions into a naturalized semantics in which reference is explained by causality, and that this requires a metaphysically substantial theory of real collections.

Reply: This is a theoretical endeavour beyond truth-conditional semantics. Why we have to engage in it demands further justification. It seems that the idea aspires to a metaphysical world view, respectively the integration of semantics in a naturalized metaphysics and semantics. This better calls for cognitive science (broadly taken): if cognitive science can account for plural usage by setting out the semantic knowledge involved in it, including our capacity for set talk and apprehension of real collections in situations, an explanation as far as cognition goes has been achieved. The rest (i.e. the demand under discussion) is a metaphysical program – and the rest might better be silence.

It is far from obvious that even a naturalized explanation of our cognitive faculties in cognitive science presupposes or needs a metaphysics of collections. An *explanation* of our cognitive faculties with respect to plural usage might as well relate to social patterns of linguistic usage (i.e. be in part social science or sociolinguistics), and our habits of

ontological pretence (i.e. be in part psychology or linguistic pragmatics). This explanation might include how we *perceive* real collections or group objects into collections. Such explanations might be internal to the mind (i.e. to information and representation processing in the mind) and *reference* to perceptual input in its causes may play no role at all, let alone a causal theory of reference involving a metaphysics of causal antecedents. The demand for a substantial metaphysics is just that: a metaphysical program, which – as always – carries the burden to prove its decisive role and usefulness. The position set out here claims that semantics might be embedded into a cognitive science account of our faculties, but this does not extend to a bridge into metaphysical world-views and general programs.

§12 Some have entertained a logic of plurals as an *alternative* to an ontology of sets. Reply: Given either (i) a commitment to sets in some (mathematical) parts of language anyway or (ii) a fictionalist attitude towards general trans-finite set theory, this avoidance of sets in logical semantics and FOL quantification will not be an issue of importance. An issue could only be whether one wants an ontology going beyond sets.

A reduction or elimination of sets in favour of plurals should be doubted because the iterative use of sets in *V* cannot be modelled by an iterated use of ever higher pluralities. Natural language use seldomly contains expressions for even 2^{nd} order pluralities (e.g. a league is a collection of clubs which are collections of players).

§13 The key controversy puts the alternative centre stage of using (i) FTZU – or if need be whatever higher-order or intensional logic already found suitable for natural language – as language and logic or (ii) not using sets as referents for plurals but extending the logic of natural language with a *logic of plurals*. Option (i) pleads for ontological and logical simplicity, option (ii) pleads that explanatory adequacy with respect to plural talk demands mor logical and, maybe, more ontological resources.

§14 Ordinary language employs “set” in non-mathematical contexts (“He owns a set of cows”) alongside plural expressions (“His collection has been expensive”). There are no strong or even knock-down arguments starting from ordinary usage to eliminate either set or plural talk. The proponent of a set *semantics* of plurals does not aim at eliminating plural terms, she only aims at avoiding a new type of entity introduced into the semantics of plural talk.

§15 Arguments for a new PQL with a corresponding new semantics for plurals have often hinted at sentences which, they argue, pose problems for FTZU, because no obvious reference to a set appears at the surface. For instance:

(4) Paul and Paula are opera singers, and they delivered a tremendous duet.

The “they” seems to need a collective reading, and introducing a semantic reference to a set here seems artificial, the 1st part of the sentence talks about them individually.

Reply: (i) ‘artificial’ is not equivalent to ‘wrong’, (ii) the main question of sentences like (4) is where to put the plurality: an alternative to a collective reading of “they” is to *put it into the semantics of the relation term* “duet”. Duets are performed together by those participating in them as a form of cooperation, this cooperation being between individuals. Thus,

(4’) $O(p_1) \wedge O(p_2) \wedge D(p_1, p_2)$

will – at some coarse-grained level of analysis – be good enough for (4). The debate of what is involved in singing a duet (the truth-conditions of some event *being a duet*) have to be explained by any semantics in detail, be it of the type PQL or FTZU.

There seems to be no obstacle to follow this strategy for other predicates and relation terms which imply cooperation. This may well be the paradigm strategy and answer to criticism of set-based accounts of plurals. The *phenomena* of cooperation and interaction are aspects of some predicates and relation terms. They are *not hidden* in special plural quantifiers.

§16 This strategy can also be applied to 'mixed sentences', which contain a distributive and a collective use of plurals, like:

(5) Paula and Paul met shortly before their duet and changed apparel.

Each of them (individually) changed their clothes. The 2nd part can only be read in a distributive sense. In the 1st part "met" requires a collection, so this can only be read in a collective sense. It is within the semantics of "meet" that any supposed tension resolves: "meet" in its lexical entry is explained, *inter alia*, by the thematic roles of *attendees* of a meeting, and one is an attendee individually. So, in (5) not even sets need to be present in a (coarse-grained) logical form, like:

(5') $(\exists x, y)(x = p_1 \wedge y = p_2 \wedge M(x, y) \wedge C(x) \wedge C(y))$

§17 In linguistics conjoining of Nominal Phrases [NP and NP] is commonly analysed as a form of joining two NPs referring to individuals into a unit which need not be read collectively.

(6a) Peter and Paula spent their holidays a continent apart.

Such conjoining rules also apply to Verb Phrases, like:

(6b) She sang and watched the audience.

which does not mean that the actions referred have to be in any form collectively, or simultaneously carried out. They need not have any connection:

(6c) She was American and inherited a fortune.

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So, there no need for PQL to analyse the logic of plurals, *set theory is enough*, if not more suitable in the first place.