

Brady's Universal Logic Programme

Ross Brady, a Relevant Logics theorist and a co-worker of Richard Routley, has developed a Universal Logic approach over many years. This exposition focusses on the state of his theory in his paradigm book *Universal Logic* (2006).

- Brady's Universal Logic is a logic of **entailment** (as the main conditional). The semantics of entailment is based on a theory of meaning. So, one part of this Universal Logic is to capture the proper logic of a relevant conditional, as it, supposedly, is at work also in natural language.
- The logic is also universal in the sense that one of its main tenets is to **capture naive semantics and set/class theory** despite of the paradoxes.
- Classic FOL reasoning might be **recaptured in classical contexts**. The formalization of Brady's logic distinguishes between sentences (p, q ...) and 'classical sentences' (p', q' ...). Thus, in setting up a theory or an argument we have to know **beforehand**, which sentences are classical, as these allow for classical inferences – in terms of truth preservation – otherwise not valid in the Universal Logic.
- The **meta-theory** of Brady's Universal Logic is **classical**! Brady does not follow the 'strong universalist programme' of semantic closure and dropping the distinction between object-logic and meta-logic. Within this classical meta-theory Brady shows that his class/set theory is not trivial (i.e. is absolutely consistent).

Criticism of Classical Logic

- Brady's criticism of classical PC starts with the paradoxes of Material Implication. Whereas the following sentences may be acceptable for a truth preserving conditional, they are rather anomalous if the conditional is to express a stronger connection, as our ordinary conditional seems to do.

$$p \supset (q \supset p)$$

$$\neg p \supset (p \supset q)$$

$$(p \supset q) \vee (q \supset p)$$

$$(p \supset q) \wedge (r \supset s) \supset (p \supset s) \vee (r \supset q)$$

$$(p \wedge q \supset r) \supset (p \supset r) \vee (q \supset r)$$

For instance, in the last case, a conjunctive antecedent should express that **both** conjuncts are needed for the consequence. The truth table for $\supset$, however, makes this a logical truth. The logic of $\supset$ thus does not allow to read valid principles of conditional reasoning of the logical truths.

- The conditional of classical modal logics ($\Box(p \supset q)$ or $(p \rightarrow q)$) inherits the main paradoxes of implication like the first two.
- To capture conditional inferences a better conditional connective has to be incorporated into our logic.
- Brady, nevertheless, does *not* primarily aim at a theory of the ‘ordinary’ conditional of natural discourse with all its (e.g. pragmatic) intricacies and subfields (like theories of counterfactuals), but at a theory of the **proper conditional for mathematical and logical theories**. Despite these theories often being considered extensional they need a stronger conditional than $\supset$, namely entailment $\rightarrow$. The semantics for $\rightarrow$, however, is founded on reflections how “to formalize natural language logical concepts” (34) in the context of a theory of meaning relations (see below).
- The **Relevance Condition** (of sharing a propositional letter between antecedent and consequent) is a *step* towards securing a content connection expressed by a conditional. Entailment goes beyond this condition: the theorems for $\rightarrow$ should also **capture our intuitive ideas of valid theorems of our respective logical theories**, i.e. the logic of $\rightarrow$ should not be too weak. Universal Logic should be suitable for logical practice.

Brady’s Universal Logic is the logic DJdQ. “Q” stands for “quantificational”. Quantification is less an issue between Relevant Logics and FOL (i.e. they basically agree on the semantics of the quantifiers) than the differences for propositional connectives. DJ is a Relevant Logic.

DJdQ has the connectives: $\neg$ [Non-Boolean Negation], $\vee$, $\wedge$, $\rightarrow$ [Entailment]

Other connectives can be defined by them, e.g. $A \leftrightarrow B \stackrel{\text{def}}{=} A \rightarrow B \wedge B \rightarrow A$

Starting from the usual definitions of formula, use of brackets etc. the logic can be stated as the following axiomatic system:

Axioms

1. $A \rightarrow A$
2. $A \wedge B \rightarrow A$
3. $A \wedge B \rightarrow B$
4. $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$
5. $A \rightarrow A \vee B$
6. $B \rightarrow A \vee B$
7. $(A \rightarrow B) \wedge (C \rightarrow B) \rightarrow (A \vee C \rightarrow B)$
8. $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$
9. $\neg\neg A \rightarrow A$
10. $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$
11. $(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C)$
12. $\forall x A \rightarrow A(y/x)$ where y is free for x in A
13. $\forall x(A \rightarrow B) \rightarrow (A \rightarrow \forall x B)$ where x is not free in A
14. $\forall x(A \vee B) \rightarrow A \vee \forall x B$ where x is not free in A
15. $A(y/x) \rightarrow \exists x A$ where y is free for x in A
16. $\forall x(A \rightarrow B) \rightarrow (\exists x A \rightarrow B)$ where x is not free in B
17. $A \wedge \exists x B \rightarrow \exists x(A \wedge B)$ where x is not free in A

Rules

1. $\vdash A, \vdash A \rightarrow B \Rightarrow \vdash B$
2. $\vdash A, \vdash B \Rightarrow \vdash A \wedge B$
3. $\vdash A \rightarrow B, \vdash C \rightarrow D \Rightarrow \vdash (B \rightarrow C) \rightarrow (A \rightarrow D)$
4. $\vdash A \Rightarrow \vdash \forall x A$

Meta-Rules

1. If $\vdash A \Rightarrow \vdash B$ (without using R4 on x free in A) then $\vdash C \vee A \Rightarrow \vdash C \vee B$
2. If $\vdash A \Rightarrow \vdash B$ (without using R4 on x free in A) then $\vdash \exists x A \Rightarrow \vdash \exists x B$

Rules (like *Modus Ponens* [R1]) state how to derive theorems from other theorems (including the axioms).

A Meta-Rule states how to derive theorems from other theorems given some previous derivation. The disjunctive character of the Meta-Rules accounts for the “d” in “DJdQ”.

Meta-Rules derive rules (not theorems directly): look at the consequent of MR1.

Initially $A \vee \neg A$ is **not** an axiom.

- The logic has way more axioms and rules than standard systems as the logic is a Relevant Logic based on relevant entailment, and not on $\supset$.
- DJdQ can be presented also as a system of **natural deduction**. One main restriction consists in requiring that **a premise to be discharged has been used**. The system can use the usual rules of introducing and eliminating connectives [like $(\rightarrow I)$, $(\rightarrow E)$, $(\wedge I)$, $(\wedge E)$] with the addition of a rule of DeMorgan-Distribution (see Axiom 8) and dropping the classical rule of $(\neg I)$, which leads to Explosion (see below). $(\neg E)$ is fashioned as: derive $\neg A$ from $\neg B$ and $A \rightarrow B$ (given the restrictions of premise use). The quantificational rules add the standard restrictions on generalization [for $(\forall I)$], and closely transpose into rules the Axioms 12 – 17. The Meta-Rules are transposed into modified Dilemma and $(\exists E)$ rules requiring that the conditionals to be used are all part of the same proof (i.e. are not further nested in a sub-proof).
- Many of the **theorems** of FOL can be derived in DJdQ (with $\rightarrow$ replacing $\supset$).

These include the distribution of quantifiers, the typical theorems for $\wedge$ and $\vee$, as well as:

$$(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A)$$

$$\neg\neg A \leftrightarrow A$$

i.e. typical negation theorems like Contraposition and Double Negation Elimination, although TND does not hold.

Single substitution of equivalences in a context formula is also derivable (and thus permissible):

$$\vdash A \leftrightarrow B \Rightarrow \vdash \dots A \dots \leftrightarrow \dots B \dots$$

Addition of Classical Formula

For **classical recapture** (capturing all classical inferences in consistent contexts) classical formulae A' , B' etc. can be added, for a system DJdcQ (“c” for “classical”). For these formulae one can add

$$\text{Axiom 18} \quad A' \vee \neg A'$$

$$\text{Rule 5} \quad \vdash \neg A', \vdash A' \vee B' \Rightarrow \vdash B'$$

$$\text{resp. } \vdash A', \vdash A' \supset B' \Rightarrow \vdash B'$$

A formula is classical iff it is bivalent, and TND holds. For classical formulae $[\neg]$ comes down to Boolean Negation.

Explosion then also is recaptured for classical formulae.

$$\vdash A', \vdash \neg A' \Rightarrow \vdash B$$

Thus, the theory of classes and sets involving the paradoxes must not use classical formulae.

Properties of DJdQ

DJdQ has (almost) the standard useful properties of a logic system. Some of these are established with respect to its non-standard possible worlds semantics.

Important proof theoretic properties distinguish between truth preservation in ‘regular worlds’, in which all logical truths hold, and whatever might be in the irregular worlds. If we assume that the actual world is regular, we can be content with a meta-property holding just in the regular worlds.

- (P1) (i) All (derived) rules of DJdQ are truth preserving at regular worlds.
 (ii) All (derived) rules of DJdQ are **truth preserving** in light of the semantics of entailment.

(P2) All truth preserving rules **can be derived** in DJdQ.

The first property amounts to a form of **correctness**, the second to a form of **strong deductive completeness**.

(P3) The **Deduction Theorem** holds for DjdQ: if $\vdash A \Rightarrow \vdash B$ then $\vdash A \rightarrow B$

on the presumption that A (which may be a conjunction of premisses) has been used in the derivation (resp. all premisses have been used): no vacuous introduction of $[\rightarrow]$.

Avoiding vacuous discharge is a tenet of Relevant Logics to avoid forms of

$$\vdash A \Rightarrow \vdash (B \rightarrow A)$$

(P4) DJdQ is **prime**: $\vdash A \vee B$ requires $\vdash A$ or $\vdash B$

i.e. no TND-like (vacuously true) disjunctions are provable.

Although – as we have discussed – the Disjunctive Syllogism cannot simply hold, it can hold for theorems:

(P5) $\vdash \neg A, \vdash A \vee B \Rightarrow \vdash B$

DJdQ **fulfils the Relevance Condition** of variable sharing between antecedent and consequent of an entailment:

(P6) $\vdash A \rightarrow B$ only if A and B share a propositional variable (at some depth)

DJdQ also has some surprising and – in the light of our supposed understanding of entailment – controversial properties, like

(P7) $\not\vdash \neg(A \rightarrow B)$

for any sentences A and B. One cannot prove any denial of an entailment statement!

DJdQ is based on a semantics of **meaning containment**.

Brady's Logic of Entailment

Ross Brady proposes the system DJdQ as a Relevant logic that captures the intuitive notion of entailment. Its semantics is modelled on the idea of *containment of content*. The basic idea is that a content is connected to each sentence and the relation of entailment is that of containment of content.

Brady's logic is important, since he considers it to be a universal logic that can be employed in any context. He uses it for his version of paraconsistent set theory.

Brady's logic is very weak and has to assume axioms for relations derivable otherwise.

Brady's logic does *neither* use the ternary accessibility relation *nor* the Routley star *. That, too, makes it worth looking at.

- Once again, the system DJ^dQ (in a slightly different notation)

Axiom schemes:

- (A1) $A \rightarrow A$
- (A2) $A \wedge B \rightarrow A$
- (A3) $A \wedge B \rightarrow B$
- (A4) $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$
- (A5) $A \rightarrow A \vee B$
- (A6) $B \rightarrow A \vee B$
- (A7) $(A \rightarrow B) \wedge (C \rightarrow B) \rightarrow (A \vee C \rightarrow B)$
- (A8) $A \wedge (B \vee C) \rightarrow A \wedge B \vee A \wedge C$
- (A9) $\neg \neg A \rightarrow A$
- (A10) $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$
- (A11) $(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C)$
- (A12) $(\forall x)P(x) \rightarrow P(a)$
- (A13) $(\forall x)(A \rightarrow P(x)) \rightarrow (A \rightarrow (\forall x)P(x)) *$
- (A14) $(\forall x)(A \vee P(x)) \rightarrow (A \vee (\forall x)P(x)) *$
- (A16) $P(a) \rightarrow (\exists x)P(x)$

$$(A17) \quad (\forall x)(P(x) \rightarrow A) \rightarrow ((\exists x)P(x) \rightarrow A) *$$

$$(A18) \quad A \wedge (\exists x)P(x) \rightarrow (\exists x)(P(x) \wedge A) * \quad * [x \text{ not free in } A]$$

Rules:

$$(R1) \quad \vdash A \rightarrow B, \vdash A \Rightarrow \vdash B$$

$$(R2) \quad \vdash A, \vdash B \Rightarrow \vdash A \wedge B$$

$$(R3) \quad \vdash A \rightarrow B, \vdash C \rightarrow D \Rightarrow \vdash (B \rightarrow C) \rightarrow (A \rightarrow D)$$

$$(R4) \quad \vdash A \Rightarrow \vdash (\forall x)A$$

Meta-Rules:

$$(MR1) \quad \text{If } \vdash A \Rightarrow \vdash B \text{ then also } \vdash A \vee C \Rightarrow \vdash B \vee C$$

$$(MR2) \quad \text{If } \vdash A \Rightarrow \vdash B \text{ then } \vdash (\exists x)A \Rightarrow \vdash (\exists x)B$$

where in both meta-rules in the derivation $\vdash A \Rightarrow \vdash B$ (R4) does not generalize on a free variable in A.

- For **DJ^dQ**-semantics we start with **contents**:

The content of a set of sentences Γ are all the sentences that can be analytically established from the terms occurring in the sentences in Γ .

Often, we take Γ to be a singleton $\{A\}$.

Thus, the content of a sentence comprehends everything that follows alone because of its meaning from that sentence, independently of the state of the world, i.e. all that is analytic with respect to that sentence. We need not to know exactly what meanings are, we need only to hold that some truths are truths of meaning (analytic truths) and others are empirical. So “Cora is a cat” content wise entails “Cora is an animal”, but not “Cora chases the red laser dot”.

The range of Γ is the set of all sentences that analytically ensure that at least one sentence in Γ is true.

Thus, the range of a singleton $\{A\}$ contains the preconditions that ensure analytically that A is true. If A is “Cora is an animal”, then “Cora is a cat” belongs to the range of A , but also “Cora is cow”, “Cora is a parrot” etc.

Both concepts, ‘content’ and ‘range’, presuppose that we know of a method to analytically establish consequences. We have to have at least some basic theory of analyticity.

- Each sentence A corresponds a set of sentences $c(A)$, which is its content (the analytic closure of A). The content set of A can be taken as a great conjunction.

$A \models B$ for $B \in c(A)$ is analytic.

Each sentence corresponds a set of sentences $r(A)$, which is its range. The range set of A can be taken as a great disjunction – think of the example above with all the different animals.

- A content is closed under containment with conditions like self-sufficiency [$c(c(A)) = c(A)$], self-containment [$A \in c(A)$] and monotonicity [$x \subseteq y$ implies $c(x) \subseteq c(y)$].

- The logical operations can be semantically modelled by algebraic operations on these (content) sets. Every basic sentence is assigned some content.

The content of $A \wedge B$ is the set of analytic consequences of the union of $c(A)$ and $c(B)$, $c(A) \cup c(B)$, not that union itself.

The content of $A \vee B$ is $c(A) \cap c(B)$.

Entailment is now semantical defined as *containment of content*:

$$v(A \rightarrow B) = 1 \text{ iff } c(B) \subseteq c(A).$$

- To ensure a Relevant logic further postulates for content relations are laid down, e.g.

$$c(A \vee B) = c(A) \cap c(B) \subseteq c(A).$$

This states that the content of the disjunction has to be in the content of each disjunct, i.e. cannot depend on just one disjunct. We have seen that DJdQ is *prime*.

- An **exception to this algebra is negation**, since taking negation as *set complement* (like in standard Classical Semantics) would give

$$A \wedge \neg A$$

universal content, and would validate Explosion and *verum ex quodlibet* for entailment.

The negation of a sentence can be better understood as all those sentences that represent *alternative* situations, i.e. the content of $\neg A$ are all facts **the absence of which made A possible**:

$$c(\neg A) \stackrel{\text{def}}{=} \{B \mid \neg B \in r(A)\}.$$

$\neg B \in r(A)$ means – given the definition of ‘range’ – that B has to be false as one precondition to establish A *analytically*.

This set is $c(A)^*$, the complement set to a sentence A. One can prove $c(c(A)^*)^* = c(A)$: the sentences making A false analytically being false analytically themselves establishes A being true analytically – therefore Axiom (9).

The interpretation v assigns $c(A)^*$ to $\neg A$.

Negation, therefore, becomes *intensional*!

- The semantics of entailment – taking it as content containment – seems to be close to our natural understanding of entailment. Nothing about accessing the accessibility of worlds is involved.

This semantics – obviously – *presupposes* a theory of establishing the analytic consequences of a given sentence, i.e. a theory of analyticity! This may require **meaning postulates**, which in turn may contain expressions (like $[\rightarrow]$) the semantics of which operates on contents! A meaning postulate – usually a conditional – may look like the prototypical entailment. *Basic* content assignment (at the beginning of the interpretation semantics) therefore may not be so basic at all.

- Notwithstanding this, Brady's logic seems to have some nice meta-logical properties:

DJdQ is sound and complete relative to the semantics of contents. Sentences with identical contents can always be *substituted* for each other.

[For contained meaning semi-substitutivity in a context holds.]

- As we have seen, several typical theorems do not hold in **DJ^dQ**:

- Contraction $[(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)]$

Even if the content of $A \rightarrow B$ is contained in the content of A – whatever that means – this *fact itself* does **not analytically imply** $A \rightarrow B$. The stress is on ‘analytically imply’ as this the truth condition of an entailment $[\rightarrow]$. Nested $[\rightarrow]$ are difficult to establish and understand in the entailment semantics. Often DJdQ may have (derived) rules on preservation of (logical) truth instead of theorems exhibiting nested entailments.

- The Law of Excluded Middle (TND) $[A \vee \neg A]$

The content of A $c(A)$ and $c(A)^*$ combined **need not exhaust** all possibilities as $c(A)^*$ is not the *set complement* of $c(A)$, as seen above.

- Disjunctive Syllogism $[\vdash \neg A, \vdash A \vee B \Rightarrow \vdash B]$

- The *Modus Ponens* Theorem $[A \wedge (A \rightarrow B) \rightarrow B]$

The content of $c(A) \cup c(A \rightarrow B)$ needs not *analytically imply* B , even though $\vdash A, \vdash A \rightarrow B \Rightarrow \vdash B$ – the *proof* of the preconditions allows logical closure

- Negation Introduction $(A \rightarrow \neg A) \rightarrow \neg A$.

Even if $c(A)$ contains $c(A)^*$ – which hardly makes sense – this fact itself does contain $c(A)^*$.

- In contrast, if we go through the list of **axiom schemes they can be justified** in terms of the semantics of entailment.

[The rules need only be justified as preserving (logical) truth, which they clearly do.]

(A1) $A \rightarrow A$

Obviously, A can be analytically be established from A.

(A2) $A \wedge B \rightarrow A$

(A3) $A \wedge B \rightarrow B$

The union of contents $c(A)$, $c(B)$ contains $c(A)$ and $c(B)$

(A4) $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$

If $c(B) \subseteq c(A)$ and $c(C) \subseteq c(A)$ it follows (analytically) that $c(B) \cup c(C) \subseteq c(A)$ by set theory.

(A5) $A \rightarrow A \vee B$

(A6) $B \rightarrow A \vee B$

The content of $A \vee B = c(A) \cup c(B)$ is a *subset* of the content of the disjuncts.

(A7) $(A \rightarrow B) \wedge (C \rightarrow B) \rightarrow (A \vee C \rightarrow B)$

If $c(B) \subseteq c(A)$ and $c(B) \subseteq c(C)$ it follows (analytically) that $c(B) \subseteq c(A) \cap c(C)$ by set theory.

(A8) $A \wedge (B \vee C) \rightarrow A \wedge B \vee A \wedge C$

Corresponds to set theoretic analytic statement:

$c(A) \cup (c(B) \cap c(C)) \subseteq c(A) \cap c(B) \cup c(A) \cap c(C)$.

(A9) $\neg \neg A \rightarrow A$

By definition: $c(\neg \neg A) = \{B \mid \neg B \in r(\neg A)\}$ i.e. the set of all sentences which have to be false to analytically establish $\neg A$. $c(A)$ is the set of sentences analytically implied by A. $c(A) \subseteq c(\neg \neg A)$, because A analytically implies that $\neg A$ is false.

[Remember that Brady does not endorse explicit contradictions $A \wedge \neg A$.]

(A10) $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$

If the sentences that analytically exclude B are part of the content of A, then B being a fact analytically establishes that A cannot be the case.

(A11) $(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C)$

Corresponds to transitivity of subsets:

$c(C) \subseteq c(A)$ by set theoretical analytic implication from $c(B) \subseteq c(A)$ and $c(C) \subseteq c(B)$.

(A12) $(\forall x)P(x) \rightarrow P(a)$

The meaning of $[\forall x]$ clearly implies the truth of an instance.

$$(A13) \quad (\forall x)(A \rightarrow P(x)) \rightarrow (A \rightarrow (\forall x)P(x))$$

This and (A14), (A17), (A18) hold by the meaning of vacuous quantification: If for all objects the meaning of A analytically implies that P is true of that object where A does not say anything of that object (“x” is not free in A), then the meaning of A entails that P is true of all objects.

$$(A14) \quad (\forall x)(A \vee P(x)) \rightarrow (A \vee (\forall x)P(x))$$

$$(A16) \quad P(a) \rightarrow (\exists x)P(x)$$

By the meaning of $[\exists x]$.

$$(A17) \quad (\forall x)(P(x) \rightarrow A) \rightarrow ((\exists x)P(x) \rightarrow A)$$

$$(A18) \quad A \wedge (\exists x)P(x) \rightarrow (\exists x)(P(x) \wedge A)$$

Set Theory and Class Theory

Routley and Brady proved the non-triviality of a paraconsistent set theory using a logic with a negation semantics in terms of the Routley * and the ternary accessibility relation.

Brady improved on that situation by stating his set theory in his logic DJ^dQ.

Overview. Brady's Paraconsistent Class Theory

Brady's version of paraconsistent set theory like Routley's DST employs Naive Comprehension (NC1) – but *with* the restriction that the class y to be determined does not occur in the determining predicate – and Extensionality in its rule version. The underlying logic is DJ^dQ.

Given the semantics of content containment Brady has to use the **rule form of extensionality**, since the *content* of $x = y$ seems not to contain $x \in z \leftrightarrow y \in z$.

In fact, Brady's theory is a *class theory* (not a set theory).

Brady **distinguishes classes** for which the two axioms hold and for which sentences dealing with them have a Relevant logic (namely DJ^dQ) **from sets**, whose memberships sentences obey classical standard logic! The standard behaviour is needed to have enough countable sets in the classes. **On the sets we thus have a completely standard logic and ontology.**

The interesting part now is the class theory. Brady has proved his system of class theory to be non-trivial.

In fact, Brady tries to avoid the antinomies in the first place, not just their spreading of inconsistency! Brady's version of paraconsistent set theory does *not* contain all of the antinomies and 'only' keeps them from spreading triviality elsewhere. **Some of the antinomies do not occur.** In case of the Russell set R one can prove $R \in R \leftrightarrow R \notin R$. To get to the explicit contradiction $R \in R \wedge R \notin R$ one needs either the Law of the Excluded Middle (TND) or Negation Introduction. Both are absent in DJ^dQ . Thus, given the validity of $R \in R \leftrightarrow R \notin R$ only one can choose to **make them both true or both false**. Both cases are harmless with respect to the behaviour of the Russell set, but this **shows some semantic underdetermination** of Brady's theory.

Something similar holds for Curry's Paradox, since Contraction does not hold in DJ^dQ .

Since Brady distinguishes sets from classes, he **restricts the validity of *Cantor's Theorem*** (that the powerset of a set has more elements than the set itself) **to sets**, avoiding the antinomy that the powerset of the universal set has to be within the universal set and at the same time larger than the universal set. (The sets are collected into a class.)

Cantor's Theorem

Cantor's Theorem states that the power set of a set has a greater cardinality than the set itself.

Brady restricts its validity in his paraconsistent class theory. The standard proof of *Cantor's Theorem* does not go through in $DJdQ$ because **it requires TND**. Therefore, it does not hold for *classes*, the logic of which is pure $DJdQ$, but for sets, as for classical formulae – as an extension of $DJdQ$ – TND holds and sets are (by Brady's definition) classical.

The standard proof shows that there is no 1:1 correspondence between the elements of some set x and the elements of its powerset $\wp(x)$. If there was such a 1:1 correspondence each element $z \in x$ would have to have one and only one corresponding subset $y \subseteq x$. Call this correspondence f . One defines that subset y' containing all $z \in x$ which are not in the subset of x corresponding to them according to f . $y' = \{z \in x \mid z \notin f(z)\}$.

If there was f , there would be a z' such that $y' = f(z')$.

But this leads to contradiction $z' \notin y' \rightarrow z' \in y'$, and vice versa: $z' \notin y' \leftrightarrow z' \in y'$

Now, in DJdQ nothing contradictory follows from this, especially not that there is no such correspondence f .

That the standard argument does not go through does not show that the powerset of a set *is never* of greater cardinality than the set itself.

There may be other procedures to show the greater cardinality of some set compared to another (e.g. a grid which by graphic diagonalisation shows that some set cannot be enumerable).

What this example shows, however, is that it is – by far – not clear that all standard theorems have corresponding theorems and proofs in a paraconsistent setting.

Giving up *Cantor's Theorem* has immense effects not only in set theory, but also in philosophical meta-logic. Grim argues by *Cantor's Theorem* that the universe is incomplete, not containing a set of truths, facts, propositions ...

Whether being expelled from ‘Cantor's paradise’ of ever higher infinite cardinalities is a real loss to mathematics and set theory may be doubted, however.

Some Details

- **Classes are determined by natural language predicates.** They are extensions of predicates.

If (NC1) is phrased with $[\rightarrow/\leftrightarrow]$

$$\exists y \forall x (x \in y \leftrightarrow \varphi(x))$$

then this stresses that the class is **defined by its corresponding predicate** (i.e. a **meaning** relationship).

- The defining predicates can contain membership statements, but these have to be **grounded** to avoid circularity in membership determination. ‘Groundedness’ means building first collections from individuals and $\perp$, and then build further collections step by step, somewhat *like* in the iterative hierarchy of standard set theory ZFC. Classicality and groundedness are the defining conditions of **sets**. **Sets are arbitrary collections** (i.e. their members *need not* have a property determined by a predicate in common – apart from the trivial property of being a member of that set). **Sets obey the usual axioms of ZFC**. Brady even adds an Axiom of Inaccessibility (like an axiom for large cardinals) but not the *Generalized Continuum Hypothesis*.

[This leads to the curious side effect that the null *set* and the null *class* will differ!]

- The collection G of all grounded sets cannot be a grounded set on pains of the contradiction

$$G \in G \in G \in G \dots$$

showing the ungroundedness of G . So, G has to be a class.

- **Cardinals and ordinals are set up *sui generis*** – as the number of predicates, and thus of classes, is only countably infinite, and the sets of the theory here are not exactly the ZFC sets (e.g. by the logic DJdQ-framework being intensional).

- Sentences like

$$R \in R$$

for the Russell set/class R , or the Liar, or similar antinomies, are considered ‘indeterminable’ – having no TND – the truth value of which can be assigned at each possible world **arbitrarily**.

Brady’s theory thus **allows for** some intuitions concerning ‘class’ to be realized in a theory (most importantly Naive Comprehension) that also allows for arbitrary collections (here: ‘sets’) as well.

Although the paradoxes can be formulated, **explicit contradictions are avoided**.

He proves this theory to be non-trivial (non-explosive): not $(\vdash A \text{ and } \vdash \neg A)$, although it may be that $\vdash A \leftrightarrow \neg A$. But even from $\vdash A \leftrightarrow \neg A \not\Rightarrow \vdash B$ for just any B .

On the other hand, the theory may be **seen critical**, given its

- rejecting TND
- having negation as an intensional connective
- having Extensionality only in rule form to fulfil expectations of relevance (like the Relevance Condition)
- introducing sets like the sets of ZFC, but forsaking the idea of the Iterative Hierarchy, which provides a natural picture of sets and their ranks

- introducing ordinals and cardinals *sui generis* instead of reducing them to sets, and members of classes cannot be simply enumerated (or well-ordered)
- having universal collections of sets (like G or the class U of all sets), but no corresponding universal collections *of classes*, raising the difficulty of how to conceive the intended domain/universe of the theory.

Criticism of Relevant Logic

- Although Brady endorses the Relevance Condition and his logic is (in the terminological sense of avoiding the irrelevant theorems of Classical Logic) a Relevant Logic, he criticises some uses and theories of Relevant Logic.
- As seen above, the **Relevance Condition by itself is too weak** to ensure a strong enough logic (e.g. containing transitivity principles: if A, B in $A \rightarrow B$ and B, C in $B \rightarrow C$ have letters in common that does not ensure that A and C in $A \rightarrow C$ have a letter in common). One has to look for a logic strong enough, but still relevant.
- Naive set theory raises some problems of relevance for the Axiom of Extensionality (expressed with $[\rightarrow]$) and substitution of identical (as Routley observed). But this is one of the logical theories to be captured. One thus cannot just – in confidence of $[\rightarrow]$ – replace all $[\supset]$ in the theories concerned by $[\rightarrow]$ and think this will work out fine.
- Most crucially one needs a principled semantic foundation for one's theory of entailment $[\rightarrow]$. Brady aims to deliver this with his theory of meaning containment. The basic idea is that **the meaning of the consequent has to be contained in the meaning of the antecedent**. This reliance on meaning does **not require a full theory what the essence of meaning is**, but a theory of **meaning relations**, what defines 'meaning containment', whatever meaning is itself. Meaning, thereby, is one of the central and basic concepts of Brady's Universal Logic. The second central semantic concept, of course, is 'truth'. And truth relations allow to recapture classical reasoning in terms of $[\supset]$.
- In comparison to some Relevant Logic like the system R, the semantics of meaning containment may lead to a – comparatively – **weaker logic** (i.e. some theorems of R are invalidated).

- Some Relevant Logic systems have extra connectives (like an intensional conjunction called “fusion”). DJdQ allows the addition of some of these (e.g. of fusion defined as ‘cotenability’ of two statements: $\neg(A \rightarrow \neg B)$), but they have a use only as a shorthand for entailment sentences.

Negation and Tertium Non Datur

- Like Routley’s Ultralogic Brady’s Universal Logic does not – except in contexts of classical recapture – contain Classical (‘Boolean’) Negation ($\sim A$ being true iff A is not). $[\neg]$ in Universal Logic is different from $[\sim]$. Classical Negation leads to paradoxes of conditionals like: $A \rightarrow B \vee \sim B$

As these are to be avoided so is the general presence of Boolean Negation, although in some contexts $[\sim]$ has a use.

- Centrally Brady’s Universal Logic **does not contain TND**, in contrast to Routley’s Ultralogic (although TND there is phrased not in terms of Boolean Negation), except, again, in contexts of classical recapture. In classical contexts: $A \vee \sim A$

TND is rejected by a reflection on the **meaning of negation**. [Doubts about this in the context of an appeal to the logic of natural language will be articulated later.]

- Brady’s treatment of the paradoxes of naive set theory and semantics relies crucially on the absence of TND. Brady blocks the step from

$$r \in r \leftrightarrow r \notin r$$

to the explicit contradiction

$$r \in r \wedge r \notin r$$

as this step requires TND (or equivalent principles). As we have seen, Routley takes this step by TND.

One point of disagreement between Routley’s and Brady’s universal logic programmes is the validity of TND. Routley endorses TND, and employs it to derive explicit contradictions from the paradoxes. Brady denies TND – outside classical recapture – indirectly for the reason to avoid just these derivations.

In the light of the strong universal logic programme, this poses the question whether our natural understanding of negation, our use of negation in natural (language) reasoning endorses TND.

Is there room for **truth value gaps** in (strong) universalist reasoning? I don't think so.

There is something wrong with the concept of truth value gaps when applied to natural languages. Many 3-valued approaches to paradox and antinomies can be rejected as insufficient, because they face strengthened versions of the antinomies, or as linguistically limited (in the sense of relying on further linguistic resources), because they employ distinctions in semantics which they cannot express themselves. They are applicable only to partial languages and not to inclusive languages like a natural language, which has to have the resources to express any semantic distinction, as the arguments introducing these distinctions are expressed in natural language or a canonical regimentation of it. So many such gap theories fall prey to 'Strengthened Liars' and 'Revenge', which reintroduce the bifurcation needed for the antinomic reasoning. This can be expanded into an argument for the inexpressibility of real gaps in natural language.

Even leaving Strengthened Liars to the side and exploiting the presumably universally applicable meta-semantic resources of ordinary language coupled with the resources of a logic dealing with defaults or retractions a gap approach violates the idea that ordinary language can express all semantic facts (crucially including those the gap theory wants to talk about).

The task of universal linguistic philosophy has to be to identify the **features of the universal framework of language**. Therefore, **universal linguistic philosophy cannot restrict itself to non-universal languages**. The language of such a philosophy has to be semantically closed (dealing with its own semantics), thus – at least *prima facie* – engendering semantic antinomies! Universal theories of meaning, truth, knowledge etc. were not to have if we can talk only from some meta-language 'down' to some distinct object-language. But these are the very theories that philosophy is after. And notwithstanding their lip-service to hierarchy solutions of the antinomies most philosophers propose their *general* theories of meaning, truth, belief, reference, knowledge etc. They are right to do the latter, since we have such universal concepts. Natural language turns out to be *the last* meta-language, that meta-language in which the most basic formalisms of some other meta-language were introduced. And natural language turns out to be *the universal* meta-language in that all the formal

constructions and sentences of some new system can be translated (read) as ordinary sentences with some formal regimentation. **There is no extraordinary special or deviant new logic which can say something that we cannot say in (some) natural language. The natural languages we use seem to be semantically closed.** We can express the antinomies in these languages. Natural languages have the resources of naming and corresponding self-reference. Semantic properties can be expressed in natural languages. *Prima facie* natural languages, therefore, are the very paradigm of universal languages. If somebody wants to deny this, he has the burden of proof. Our ordinary conception of our language had to be seriously mistaken then!