Is Number a Sortal or Basically a Functional Concept?

Numbers are typically thought to be the prime example of mathematical entities. Whether abstract entities or not they are supposed to be collectable in respective number sets and can be quantified over as individual items. Numbers can be counted of.[1] Such considered numbers are a case of a sort. "is a number" might be considered the corresponding sortal predicate. Even arithmetical nominalist or fictionalists, who deny the existence of numbers, still agree that the concept "is a number" though actually empty is a sortal predicate.

The *non-realism* of Philip Hugly and Charles Sayward[2], however, opens up a quite different perspective on the concept of number. Hugly and Sayward deny that number talk is referential at all. According to them numerical expression do not occur essentially (i.e. without the possibility of paraphrasing) in referential positions. The basic construction according to them is "the number of". Thus "number" may be seen as basically a functional concept.

In the first part of this paper the theory of arithmetical non-realism is set out. Of special interest are arguments relying on language use and a theory of non-referential quantification.

In the second part some weaknesses of the arguments presented are considered. Although the theory may be right in that the functional aspect of the concept of number has been neglected too much, number has to be a sortal predicate as well.

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1 We consider here natural numbers only.

2 Philip Hugly and Charles Sayward, Arithmetic and Ontology. A Non-Realist Philosophy of Arithmetic. Amsterdam/New York, 2006.

I. Number as Functional Concept

What is the basic usage of "number"? Before we start theorizing about numbers and their arithmetical properties we count things.

(1) How many apples are on the table?

(2) There are 3 apples on the table.

(3) The number of apples on the table is 3.

There is a difference between this kind of applied arithmetic and pure arithmetic. Pure arithmetic consists of sentence which contains only the vocabulary of formal arithmetic (i.e. no empirical concepts).

(4) 5 = 3 + 2.

Pure arithmetic nevertheless has an empirical application in that we use such statements to make inferences about empirically observed amounts. We derive

(5) The number of items on the table is 5.

from (3) and

(6) The number of peaches on the table is 2.

with (4) and

(7) No peach is an apple.

Apart from that pure arithmetic can be considered to be a purely formal (syntactic) calculation using an axiomatization of arithmetic. Thus it would not commit us

ontologically beyond the commitment in the empirical statements of number, which, however, may be non committal.
If ontological commitment can be read off from the use of existential quantification, one may ask whether these have to be used. Concerning natural numbers, however, one may regard an existential quantification as a disjunction of countably many formulas. Each of the disjuncts would be a numerical statement. And if numerical non-compound statements are not ontologically committal existential quantification cannot be either! The standard assumption about the referential impact of existential quantification about numbers therefore turns out to be mistaken!
Two central theses have to be argued for, then:

(T1) The basic use of "number" is a functional use.

(T2) The formal apparatus of pure arithmetic employed in inference with empirical number statements does not yield further ontological commitment.

We consider the main arguments for these theses, respectively.

Argument 1

Numerical expressions are basically adjectival or can be paraphrased away

The basic context in which "number" occurs is the context "the number of" preceding a noun. (Meaning and reference should be determined only with respect to the context in which an expression usually occurs.)
Sentence

(3) The number of apples on the table is 3.

supposedly speaking of numbers can be paraphrased:

(8) There are some x, y, z: $x \neq y$, $y \neq z$, $x \neq z$, x is an apple on the table, y is an apple on the table, z is an apple on the table, and for all w: w is an apple on the table, only if $w = x$ or $w = z$ or $w = y$.

Sentence (8) no longer speaks of apples. Number talk seems to be just shorthand for individuation. Sentences comparing amounts in terms of numbers, like

(9) The number of apples on the table = the number of peaches on the table.

can be paraphrased:

(10) For every apple on the table there is a corresponding peach on the table.

using some theory of establishing/defining correspondence functions.

Argument 2

A schematic (non-referential) interpretation of pure arithmetic is possible and sufficient

The Peano/Dedekind axioms of arithmetic use the numeral "0" and quantify over numbers, for example ("s( )" denoting the successor function):

(PA1/2) $(\forall n)(0 \neq s(n)) \wedge (\forall n)(\forall m)(s(n) = s(m) \supset n = m)$.

The equations used in arithmetic reasoning can be arrived at, however, without using these axioms. One may rather use schemata, like

(PA 1/2*) Accept all sentences that result by replacing variable by *numerals* in: $0 \neq s(x) \wedge (s(y) = s(z) \supset y = z)$.

This gives us the non-quantificational equations of arithmetic. (PA 1/2*) does not talk about numbers, just about numerals. Schemata along the line of (PA 1/2*) for the six usual arithmetic axioms [without Induction] provide a (weakly) *complete* axiomatization of the set of non-quantificational arithmetical theorems.[3] Since universal quantification is left out this theory is obviously ω-incomplete, but so (by Gödel's First Incompleteness Theorem) is Peano Arithmetic (since the Gödel sentence as a universal sentence cannot be derived)[4].

The schemata and an accompanying proof procedure provide us with the needed arithmetical equations. And by reference to the proof procedure we have an understanding how pure arithmetical *knowledge* is obtained. Mathematics is about theorems not about numbers (i.e. the discipline mathematics is concerned with

[3] Substitutional accounts of quantification (similar to the usage of schemata) are neither compact nor can they be *strongly* complete, since there are not enough expressions around. Given infinitely many numerals the set of all sets of formula (considered in the quest for strong completeness) is non-denumerable.

[4] The Gödel sentence says (in one way of rendering some such sentence): For all numbers: This number is not the number of my proof. Since the proof predicate is recursive we can prove (in the system) for any number that it is not the number of the proof of the Gödel sentence, but we cannot prove the generalization itself.

theorems not with their supposed referents). One has arithmetical knowledge if one knows of some equation α that it can be proven. That some equation can be proven does not entail that it is about something. Neither does the declarative *form* of the arithmetical equations entail that they are about something. Outside of pure arithmetic these equations are *used* as rules/inference-tickets in talk about physical objects.

Argument 3

Not all quantification is referential

In sentences like (3) and (6) the numerals do not occur in overtly referential positions. Thus one may deny that they are referential. If one uses schematic pure arithmetic the question of interpreting existential quantification does not arise. Even if one considers quantificational talk about numbers one may ask whether this talk has to be taken as referential. A lot of quantificational talk does not seem to be referential, say

(11) There is more to life than mathematics.

If the instances of a existential quantification are finite in number, the quantification is just shorthand for a long disjunction. If none of the disjuncts is referentially committal neither can be the disjunction.

Even if one allows for infinitely many numerals and for existential quantification in arithmetic not being reducible to disjunction, there is no positive argument why one should take such a statement as referentially committal. If one does not presuppose (mathematical) realism one may always resort to a substitutional reading, leading us back to numeral equations. Quantification is objectual only if the bound variables are open for substituends which are referential expressions. Numerals are no such expressions.

Additional Arguments [not by Hugly and Sayward]

- If numbers were independent entities then it should be possible that there are apples but not numbers. Then it should also be possible that there are 3 apples without there being the number 3.

- Pure arithmetic may be considered as a context in which sentences occur. The statements of pure arithmetic may then be analyzed as containing a claim about arithmetic entities within the scope of a context defining operator like "In pure arithmetic: 17 is a prime number". The operator may be ontologically as uncommitted as "believes that" or "In the Bible:".

Given these arguments numbers cannot be taken as objects. Number talk is objective only in as much as numerals have occurrences in sentences which make objective assertions and have ascertainable objective truth conditions.

II. A Critique of Arithmetical Non-Realism

"Non-realism" as a new label does not really define a new position. If numerical discourse is non-referential, then numerals do not refer. So there are no numbers. So this is arithmetic anti-realism.

Since the central claim of non-realism is the non-referentiality of number discourse, one may ask what general criterion establishes that. One idea seems to be that there are clear cases of referential discourse and that the unavoidability of this kind of discourse about some topics commits us to corresponding entities. The argument thus proceeds negatively by showing that paraphrases exist which avoid these referential expressions. Not having an exhaustive list of committal expressions or an independent criterion of referentiality leaves the success of this procedure an open question.

As a critique of arithmetical non-realism the arguments given by Hugly and Sayward can be questioned. We turn to this.

ad Argument 1

- Numerals are said to belong to the same semantic type as quantifiers. This is false, since then there could not be non-wellformed transformations like

  (12*) There are some apples on the desk. Some is the number of the apples.

substituting a quantified for a numerical expression.

Thus although there are numerical quantifiers not all numerals are quantifiers, quantifiers and numerals (in general) belong to different categories.

- Any paraphrase can be read in both directions, especially so if it is claimed to preserve meaning. Therefore some follower of Frege may step from (10) to (9) ... We need additional arguments whether we should read the paraphrase as an reduction of an elucidation (of hidden ontological commitment).

ad Argument 2

There are several problems with a mere schematic rendering of pure arithmetic.

- Some statements of pure arithmetic *apply numbers to numbers*. Such statement are not employed as inference tickets in empirical number statements. Some of these statements may be generalizations about unspecified numbers (e.g. a theorem about the existence of an enumeration). If a sentence like (2) commits us to the existence of apples why does

  (13) There are 3 prime numbers between 3 and 12.

  not commit us to the existence of (prime) numbers?

- There are statements in the language of pure arithmetic independent of the Peano axioms. A schematic account of the axioms thus is insufficient to deal with all *true* statements expressed within the language of pure arithmetic. Peano arithmetic itself (i.e. without set theory) does not even entail the existence of infinitely many natural numbers, so that the claim may be related to focussing on too weak a system. (And even if – as the claim about ω-incompleteness has it – Peano Arithmetic leaves some true statements out, this does not justify leaving even more true statements out.)

- A *categorical* characterization of arithmetic requires Second Order Logic, and thus quantifies not only over individuals, but also over sets/collections of individuals. This carries further ontological commitment. And there can only be different sets (of numbers) if there are numbers.

ad Argument 3

- One may well ask whether it is not the case that every existential quantification is referential. The examples put forth for non-referential sentences which possess a referential surface structure may well all be read as committing us to corresponding kinds of entities. Given some ontological promiscuity there seems nothing strange about a kind like pieces of knowledge, quantified over in a sentence like

  (14) There are many things I have to learn about prime numbers.

Some of these ontological commitments may be quite innocent, others may be respectable as being easily integrated into a naturalistic ontology or one that is already committed to the existence of abstract entities.
Further on, even if one questions the theory that ontological commitment can be read off from our existential quantifications, that may be for the reason that *non-quantificational* elements carry ontological commitment as well. In some versions of standard semantic analysis the only referring expression in

(15) The keyboard I'm typing on right now is black.

is "the keyboard I'm typing on right now". But seen from the perspective of truth-maker theories this is insufficient. Some structure of the world has to be there to make the sentence true if it is true. "is black" refers to some part/structure of the space-time-region identified as "the keyboard I'm typing on right now". One may call this a trope, an instantiated universal, an individual property... Applied to our argument this yields the question whether there should not be something objective in the world to make a sentence like (2) true. Such a structural universal makes it true that there are 3 and not 5 apples on the table. This structural element has to be integrated into an account what numbers refer to.

- Ontological commitment is not just a question of the (object) language itself, but also one of its semantics in its meta-language. The meta-languages of either arithmetic or natural language semantics employ sets/classes (see above).

Some **general problems with arithmetic non-realism** are:

- Arithmetic is not enough for empirical science, but a non-realist account of set theory and analysis seems to be far off or far more complicated.

- Like all anti-realistic positions one may wonder how mathematics can be so successful as a backbone of empirical sciences without dealing with existing structures.

One may see arithmetic non-realism as a further hint that the theory of natural numbers can be developed with less ontological commitment than usually invested in standard mathematics, like Quine (in *Set Theory and Its Logic*) showed that the theory of natural numbers itself does not need an axiom of infinity. In many empirical and everyday contexts "number" may be employed as a functional concept. Nonetheless does the full account of numbers and their properties, which justifies our confidence in number talk, use "number" as a sortal predicate. Numbers have to be taken as entities, may they be sets or entities of their own kind.

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