Fictionalism in Mathematics

The view developed here denies the very idea of 'revision of logic' (in some sense to be explained) and takes a structure like ZFC as the backbone of (pure) mathematics. Alluding to Carnap's famous *Principle of Tolerance* it claims that there is no room for tolerance in logic, but a lot of room for tolerance in (pure) mathematics.

§1

Psychological realism about logic is a realism about *representations*: rules of logic are representations, inferences are ways to process representations, logical structure structures representations. Logical realism of this kind fits into a Representational Theory of Mind. Mathematics, on the other hand, seems to come with massive ontological commitments, which carry over to logic, once its meta-theory is cast in model theory. Again one may argue that our conceptual scheme contains some basic mathematical concepts, prominently some concept of collections like extensions of concepts, sets or heaps. Completely different accounts might be given of these: like a Fregean theory of value ranges for extensions, some mereology (extensional [with Lesniewski] or intensional [with Simons]) for heaps, and some set theory (ranging from finite set theory to ZFC variants, to variants with sets and proper classes like NBG or MK, and many other versions like KP, NF and what not). In the vain of the discussion above one may ask which are the principles our ordinary concept of collection relies upon. Controversial, but supposedly obvious, candidates are Naïve Comprehension, Frege's Basic Law V, Existence of General Sums ...
One may doubt, however, whether evidence for one of the complete systems can be put forth. And one may now ask oneself how we have to take the ontological commitments that come with these systems. Therefore we turn to consider ontological anti-realism. Ontological anti-realism in mathematics turns out to be compatible with (psychological) realism about logic and basic mathematical concepts.
Any form of anti-realism in mathematics – just as any form of realism in mathematics – has to account for

(1) the meaning of mathematical language
(2) the *a priori* nature of mathematics
(3) the applicability of mathematics to reality in the empirical sciences.

Supposedly (1) poses the greater challenge for the anti-realist, than for the mathematical realist, whose challenge is (3).

§2

On an anti-realist view mathematics is *objective* by being true only by force of the conventions laid down. Mathematical truths are derivable, and nothing but derivable. An anti-realist in mathematics *identifies* truth in mathematics with derivability. Truth in pure mathematics is 'true in the story of pure mathematics', coinciding with being derivable from the axioms of pure mathematics by the rules of pure mathematics and logic alone. Thus mathematics is also *a priori* and analytic. This answers question (2) above. Difficult proofs may enlarge our knowledge and deepen our understanding of the impact of the conventions, so even analytic sentences can be subjectively surprising and be a gain in explicit knowledge.

Pure mathematical talk has *meaning* by the conventions of pure mathematics (i.e. rules of usage and recursive truth conditions). This is half of the answer to question (1) above.

This conception of mathematical truth stands in no conflict with Gödel's Incompleteness Theorems, as one may observe (following Wittgenstein) that the reasoning establishing the truth of the Gödel sentence takes place in *another* formal system than that in question, as well as claim (following Dummett) that the non-coincidence of truth and provability in some system only shows that our intuitive resources of reasoning – as employed in the meta-reasoning – are not completely formalized, and so the respective system may be extended indefinitely, or one may even (following Priest) derive the Gödel sentence for a paraconsistent system in that very system.

§3

The ontology of pure mathematics (i.e. pure set theory) supports this objective quality by providing a picture of independently existing entities warranting and corresponding to the objective mathematical truths. This realm is a *fiction* accompanying the conventions of mathematics. To answer question (1) completely with respect to reference of expressions in pure mathematics one thus adds: we are presented a story/a picture of a realm of entities which serve as substitute referents for expressions in pure mathematics the way fictional characters serve as substitute referents for their names, which means properly speaking they *do not refer at all*, but are 'mere' representations. Pure mathematics tells a story, but not a story *about* something, neither about the 'forms' of Platonism nor the 'non-existing objects' of Noneism.

Pure mathematics, however, is distinguished from other fictions (other arbitrary conventions) by its applicability in the sciences and everyday life. One may account for this – and so answer question (3) above – as follows:

I. Pure mathematics (i.e. ultimately pure set theory) consists of a linguistic structure containing both expressions (supposedly) referring to single entities as well as those (supposedly) referring to relations and properties.

II. Parts of reality (e.g. countable objects and their measurable properties) provide *a partial model* of this mathematical structure. A model in the not set theoretic sense: these parts of reality can be linked to mathematical expressions (e.g. in case of measurable extents of qualities) *and* the derivable consequences with respect to them (derived using the purely mathematical structure) *apply again* to parts of reality.

III. This homomorphism (in the non-technical sense of the consequences of the picture being a picture of the consequences) invites us to assume that those parts of the purely mathematical language we have not fixed to some part of reality may nonetheless be understood as having a model, because we believe that we just have not observed their counterparts yet, or we treat their counterparts like theoretical entities in the sciences, or we just don't care about these counterparts as long as the homomorphism stays stable on the observed counterparts. One may take these parts of pure mathematics as useful supplementary fictions. They are 'supplementary' as there is nothing in the partial model relating to them.

The fictionalist with this relates *the whole of mathematics* to reality by anchoring pure mathematics in a partial model of it. In this view there is some truths in a realist picture which sees mathematics corresponding to structures of reality. In this view non-applied (purely pure) mathematics can be tolerated by its service to applied mathematics. It shouldn't be taken as exploring a self-sustaining mathematical reality. We can even speculate that our conceptual scheme rather contains a mereological concept of collection, which is taken up and extended (e.g. by postulating the existence of an empty set) by set theory. In that case the conceptual support for our immediate judgement about set theory and its principles may *rest in (intensional) mereology*.

§4

Pure mathematical talk fulfils, thus, another function than scientific talk. Scientific sentences are true or false, including those containing mathematical elements, as these elements are linked to procedures of establishing the truth value of a sentence (e.g. by the use of measuring paradigms). Scientific language aims at the facts in describing reality. Pure mathematics, in contrast, can be seen as either 'true to the story of mathematics' (by being derivable given the axioms) or as just *telling and establishing* the background story to applied mathematics (being objective by its intersubjectively shareable character of conventionality). The language game of pure mathematics doesn't concern the description or denotation of mathematical facts.

An overall fictionalist account of pure mathematics claims that even pure mathematical sentences talking about finite cardinals do not refer to mathematical facts, not just those purely pure mathematical sentences dealing with the remote regions of *Cantor's Paradise*. Scientific or everyday assertions containing numerals (e.g. "There are 3 apples on these 2 tables") possess truth conditions or are linked to procedures of justification that do not involve *numbers*, even finite cardinals. Procedures and rules covering counting, measuring or employing a ruler and a pair of compass serve as bridge principles relating empirical sentences and observations to pure mathematics.

Thus the fictionalism outlined differs from anti-realistic constructivism, which at least maintains those parts of pure mathematics which have been constructed. The fictionalist doubts the use of the constructivist's further assumption that by carrying out steps of reasoning inside mathematics we have *supplied further entities*.

Besides fictionalist anti-realism there are more well-known forms of non-standard treatments of mathematics, many of which have a constructivist streak, often including their treatment of logic. One may, for instance, target the understanding of universal quantification. Contrary to Cantor's Domain Principle*,* which assumes the existence of a domain of values for the variables quantified over, one may understand universal quantification like a conditional (substitutional) claim: once a value (or a term) is provided the universally quantified sentence holds of it, how many whatsoever these values (or terms) may be. One need not even take the domain they are supposedly are collected from to be completely given: it might be expanding or be otherwise elusive, what counts is only the conditional claim on any values (or terms) provided. This resembles using schemata with schematic expressions instead of variables and quantifiers. Such an account trades in a non-standard use of quantification (in mathematics) for ridding itself of a fictionalist account. A fictionalist account of pure mathematics need not involve a revisionist understanding of quantification, not more than any account of fiction, as fiction in general also employs quantifiers (be it counting fairies or numbers). This fictionalist anti-realism does not 'revise' standard mathematics (or ZFC) in any sense, e.g. by changing its underlying logic. The anti-realism does not pertain to logic, but to ontology.

Identifying something as pure mathematics serves as a rigidly syntactical *indication of scope* which puts the assertive force of the involved declarative sentences into brackets. The inscription “a novel” on the title page or cover of a book informs us that we confront a work of fiction, we put what is said into the brackets of a story – almost the same applies to “a treatise in set theory”.

§5

In as much as reality only provides a partial model of mathematics reality cannot distinguish between those mathematical structures which are equivalent with respect to the descriptions and the projections covering the partial model. Therefore more than one mathematical structure can be applicable to reality, and thus be useful, and in this sense be justified.

This will be so for set theoretical differences (say in large cardinals) way beyond any direct relation to applied mathematics. Postulates of the existence of large cardinals may either be rejected as superfluous or may even be endorsed as equipping the complete mathematical structure with valuable structural properties like symmetry or non-arbitrariness. This may also be the case for the distinction between mathematical structures differing in the *cardinality* of the number classes involved (i.e. those being finite, enumerable or more than enumerable). Reality may not be – or supposedly is not, according to quantum mechanics – continuous, not even dense. Not just the rational and real numbers may be too much – even large *finite* cardinals may have no application to reality.

So, even if there is no conventionalism with respect to logic in as much as our logic faculty is concerned, there is plenty of room for *conventionalism in mathematics*. Quantifiers (of some sort) and thus some powers of counting objects are part of our logic faculty, set theory (say in the form of ZFC) almost certainly isn't. Carnap's Principle of Tolerance applies here, as well as meta-theoretical criteria of theory choice in pure mathematics (like symmetry or ease of computation).

§6

Insofar as pure mathematics (set theory) serves only as the background for applied mathematics and carries no ontological commitment by itself, we needn't be as concerned about set theoretical paradoxes and foundational problems as mathematical realists are. A story may contain unsolved puzzles or even confusions – they do not matter as long as they do not affect those parts of the story relevant to us. A novel, for example, may contain errors and confusions concerning the economy of a society depicted, which nonetheless may be irrelevant to its main plot (of character development or crime detection). In that vain foundational issues in set theory lose a lot of their interest to the mathematical pragmatist, quite contrary to the semantic and logical paradoxes, which highlight either an insufficient re-construction of our logic faculty or even inbuilt conceptual mismatch. The focus on applied mathematics as crucial explains the lacking interest of the working mathematician in pure mathematics. If there were more than lip service paying real mathematical realists there should be much more concern about the problems of set theoretical foundations (like the status of the universe V or the existence of $\{\}$). Working mathematicians by their pragmatism embody an anti-realistic attitude to mathematics.

§7

Psychological realism with respect to logic and pragmatism with respect to mathematics are compatible, as the logical realist stops at the existence axioms of pure mathematics (especially the Axiom of Infinity). Even a dose of *logicism* may be compatible with anti-realism in mathematics: it may be so that our logic faculty (i.e. our conceptual scheme with respect to logical concepts) allows for the derivation of some advanced mathematical concepts and structures. Realism with respect to logic meant that we *have* this one very logic, it does not mean that all concepts employed in that faculty have objective reality in application. As often with human cognition they only have to be good enough in our dealings with reality. Thus the concepts of our logic faculty may invite and sustain some elaborate fictions of pure set theory that underwrite pure mathematics. Again one has to separate immediately applicable mathematical talk of entities, structures and consequences from scaffolding. Realism with respect to logic rests in the fact that we possess a logic faculty that is the way it *is*. Anti-realism with respect to mathematics rests in the belief that there aren't neither pure sets nor numbers of any kind. There are no facts of the matter to be discovered about them. There are matters of fact concerning our logic faculty. Our best theory of logic systematizes them.

Manuel Bremer