Lectures on Universal Logic

Lecture 8 – Brady’s Theory of Classes and Sets

Routley and Brady proved the non-triviality of a paraconsistent set theory using a logic with a negation semantics in terms of the Routley * and the ternary accessibility relation.
Brady improved on that situation by stating his set theory in his logic DJ^dQ.

Overview. Brady's Paraconsistent Class Theory
Brady's version of paraconsistent set theory like Routley’s DST [cf. Lecture 3] employs Naive Comprehension (NC1) – but *with* the restriction that the class *y* to be determined does not occur in the determining predicate – and Extensionality in its rule version. The underlying logic is DJ^dQ.
Given the semantics of content containment Brady has to use the rule form of extensionality, since the *content* of x = y seems not to contain $x \in z \leftrightarrow y \in z$.
In fact, Brady's theory is a *class theory* (not a set theory).
Brady distinguishes classes for which the two axioms hold and for which sentences dealing with them have a Relevant logic (namely DJdQ) from sets, whose memberships sentences obey classical standard logic! The standard behaviour is needed to have enough countable sets in the classes. On the sets we thus have a completely standard logic and ontology.
The interesting part now is the class theory. Brady has proved his system of class theory to be non-trivial.
In fact, Brady tries to avoid the antinomies in the first place, not just their spreading of inconsistency! Brady's version of paraconsistent set theory does *not* contain all of the antinomies and ‘only’ keeps them from spreading triviality elsewhere. Some of the antinomies do not occur. In case of the Russell set R one can prove $R \in R \leftrightarrow R \notin R$. To get to the explicit contradiction $R \in R \wedge R \notin R$ once needs either the Law of the Excluded Middle (TND) or Negation Introduction. Both are absent in DJ^dQ [cf. Lectures 5 and 6]. Thus, given the validity of $R \in R \leftrightarrow R \notin R$ only one can chose to make them both true or both false. Both cases are harmless with respect to the behaviour of the Russell set, but this shows some semantic underdetermination of Brady's theory [cf. Lecture 7 on the problem of expressibility].
Something similar holds for Curry's Paradox, since Contraction does not hold in DJ^dQ.

Since Brady distinguishes sets from classes he restricts the validity of *Cantor's Theorem* (that the powerset of a set has more elements than the set itself) to sets, avoiding the antinomy that the powerset of the universal set has to be within the universal set and at the same time larger than the universal set. (The sets are collected into a class.)

Cantor's Theorem

Cantor's Theorem states that the power set of a set has a greater cardinality than the set itself. Brady restricts its validity in his paraconsistent class theory. The standard proof of *Cantor's Theorem* does not go through in DJdQ because it requires TND. Therefore, it does not hold for *classes*, the logic of which is pure DJdQ, but for sets, as for classical formulae – as an extension of DJdQ – TND holds [cf. Lecture 5] and sets are (by Brady's definition) classical. The standard proof shows that there is no 1:1 correspondence between the elements of some set x and the elements of its powerset $\wp(x)$. If there was such a 1:1 correspondence each element $z \in x$ would have to have one and only one corresponding subset $y \subseteq x$. Call this correspondence *f*. One defines that subset y' containing all $z \in x$ which are not in the subset of x corresponding to them according to *f*. $y' = \{z \in x \mid z \notin f(z)\}$.

If there was *f*, there would be a z' such that $y' = f(z')$.

But this leads to contradiction $z' \notin y' \rightarrow z' \in y'$, and vice versa: $z' \notin y' \leftrightarrow z' \in y'$

Now, in DJdQ nothing contradictory follows from this, especially not that there is no such correspondence *f*.

That the standard argument does not go through does not show that the powerset of a set *is never* of greater cardinality than the set itself.

There may be other procedures to show the greater cardinality of some set compared to another (e.g. a grid which by graphic diagonalisation shows that some set cannot be enumerable).

What this example shows, however, is that it is – by far – not clear that all standard theorems have corresponding theorems and proofs in a paraconsistent setting.

Giving up *Cantor's Theorem* has immense effects not only in set theory, but also in philosophical meta-logic. Grim argues by *Cantor's Theorem* that the universe is incomplete, not containing a set of truths, facts, propositions ...

Whether being expelled from 'Cantor's paradise' of ever higher infinite cardinalities is a real loss to mathematics and set theory may be doubted, however.

Some Details

• Classes are determined by natural language predicates. They are extensions of predicates. If (NC1) is phrased with [$\rightarrow$/$\leftrightarrow$]

$$\exists y\ \forall x\ (x \in y \leftrightarrow \varphi(x))$$

then this stresses that the class is defined by its corresponding predicate (i.e. a meaning relationship).

• The defining predicates can contain membership statements, but these have to be grounded to avoid circularity in membership determination. ‘Groundedness’ means building first collections from individuals and $\varnothing$, and then build further collections step by step, somewhat *like* in the iterative hierarchy of standard set theory ZFC. Classicality and groundedness are the defining conditions of sets. Sets are arbitrary collections (i.e. their members *need not* have a property determined by a predicate in common – apart from the trivial property of being a member of that set). Sets obey the usual axioms of ZFC. Brady even adds an Axiom of Inaccessibility (like an axiom for large cardinals) but not the *Generalized Continuum Hypothesis*.

[This leads to the curious side effect that the null *set* and the null *class* will differ!]

• The collection G of all grounded sets cannot be a grounded set on pains of the contradiction

$$G \in G \in G \in G \ldots$$

showing the ungroundedness of G. So, G has to be a class.

• Cardinals and ordinals are set up *sui generis* – as the number of predicates, and thus of classes, is only countably infinite, and the sets of the theory here are not exactly the ZFC sets (e.g. by the logic DJdQ-framework being intensional).

• Sentences like

$$R \in R$$

for the Russell set/class R, or the Liar, or similar antinomies, are considered ‘indeterminable’ – having no TND – the truth value of which can be assigned at each possible world arbitrarily.

Summary

Brady’s theory thus allows for some intuitions concerning ‘class’ to be realized in a theory (most importantly Naive Comprehension) that also allows for arbitrary collections (here: ‘sets’) as well.

Although the paradoxes can be formulated, explicit contradictions are avoided.

He proves this theory to be non-trivial (non-explosive): not ($\vdash$A and $\vdash\neg$A), although it may be that $\vdash$A $\leftrightarrow$ $\neg$A. But even from $\vdash$A $\leftrightarrow$ $\neg$A $\not\Rightarrow$ $\vdash$B for just any B.

On the other hand, the theory may be seen critical, given its

- rejecting TND
- having negation as an intensional connective
- having Extensionality only in rule form to fulfil expectations of relevance (like the Relevance Condition)
- introducing sets like the sets of ZFC, but forsaking the idea of the Iterative Hierarchy, which provides a natural picture of sets and their ranks
- introducing ordinals and cardinals *sui generis* instead of reducing them to sets, and members of classes cannot be simply enumerated (or well-ordered)
- having universal collections of sets (like G or the class U of all sets), but no corresponding universal collections *of classes*, raising the difficulty of how to conceive the intended domain/universe of the theory.