# Lectures on Universal Logic

## Lecture 6 – The Logic DJdQ. Semantic Foundations

DJdQ is based on a semantics of meaning containment.

*Brady's Logic of Entailment*

Ross Brady proposes the system DJdQ as a Relevant logic that captures the intuitive notion of entailment. Its semantics is modelled on the idea of *containment of content*. The basic idea is that a content is connected to each sentence and the relation of entailment is that of containment of content.

Brady's logic is important, since he considers it to be a universal logic that can be employed in any context [see Lecture 4]. He uses it for his version of paraconsistent set theory [see Lecture 8].

Brady's logic is very weak and has to assume axioms for relations derivable otherwise.

Brady's logic – as we have seen in Lecture 5 – does *neither* use the ternary accessibility relation *nor* the Routley star *. That, too, makes it worth looking at.

• Once again, the system $DJ^dQ$ (in a slightly different notation)

<u>Axiom schemes:</u>

(A1) $A \rightarrow A$

(A2) $A \wedge B \rightarrow A$

(A3) $A \wedge B \rightarrow B$

(A4) $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$

(A5) $A \rightarrow A \vee B$

(A6) $B \rightarrow A \vee B$

(A7) $(A \rightarrow B) \wedge (C \rightarrow B) \rightarrow (A \vee C \rightarrow B)$

(A8) $A \wedge (B \vee C) \rightarrow A \wedge B \vee A \wedge C$

(A9) $\neg \neg A \rightarrow A$

(A10) $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$

(A11) $(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C)$

(A12) $(\forall x)P(x) \rightarrow P(á)$

(A13) $(\forall x)(A \rightarrow P(x)) \rightarrow (A \rightarrow (\forall x)P(x))$ *

(A14) $(\forall x)(A \vee P(x)) \rightarrow (A \vee (\forall x)P(x))$ *

(A16) $P(á) \rightarrow (\exists x)P(x)$

(A17) $(\forall x)(P(x) \rightarrow A) \rightarrow ((\exists x)P(x) \rightarrow A)$ *

(A18) $A \wedge (\exists x)P(x) \rightarrow (\exists x)(P(x) \wedge A)$ * * [x not free in A]

Rules:

(R1) $\vdash A \rightarrow B, \vdash A \Rightarrow B$

(R2) $\vdash A, \vdash B \Rightarrow \vdash A \wedge B$

(R3) $\vdash A \rightarrow B, \vdash C \rightarrow D \Rightarrow \vdash (B \rightarrow C) \rightarrow (A \rightarrow D)$

(R4) $\vdash A \Rightarrow \vdash (\forall x)A$

Meta-Rules:

(MR1) If $\vdash A \Rightarrow \vdash B$ then also $\vdash A \vee C \Rightarrow \vdash B \vee C$

(MR2) If $\vdash A \Rightarrow \vdash B$ then $\vdash (\exists x)A \Rightarrow \vdash (\exists x)B$

where in both meta-rules in the derivation $\vdash A \Rightarrow \vdash B$ (R4) does not generalize on a free variable in A.

• For **DJ$^{d}$Q**–semantics we start with contents:

The *content* of a set of sentences Γ are all the sentences that can be *analytically established* from the terms occurring in the sentences in Γ.

Often, we take Γ to be a singleton {A}.

Thus, the content of a sentence comprehends everything that follows alone because of its meaning from that sentence, independently of the state of the world, i.e. all that is analytic with respect to that sentence. We need not to know exactly what meanings are, we need only to hold that some truths are truths of meaning (analytic truths) and others are empirical. So "Cora is a cat" content wise entails "Cora is an animal", but not "Cora chases the red laser dot".

The *range* of Γ is the set of all sentences that analytically ensure that at least one sentence in Γ is true.

Thus, the range of a singleton {A} contains the preconditions that ensure analytically that A is true. If A is “Cora is an animal”, then “Cora is a cat” belongs to the range of A, but also “Cora is cow”, “Cora is a parrot” etc.

Both concepts, ‘content’ and ‘range’, presuppose that we know of a method to analytically establish consequences. We have to have at least some basic theory of analyticity.

• Each sentence A corresponds a set of sentences c(A), which is its content (the analytic closure of A). The content set of A can be taken as a great conjunction.

$A \vDash B$ for $B \in c(A)$ is analytic.

Each sentence corresponds a set of sentences r(A), which is its range. The range set of A can be taken as a great disjunction – think of the example above with all the different animals.

• A content is closed under containment with conditions like self-sufficiency [$c(c(A)) = c(A)$], self-containment [$A \in c(A)$] and monotonicity [$x \subseteq y$ implies $c(x) \subseteq c(y)$].

• The logical operations can be semantically modelled by algebraic operations on these (content) sets. Every basic sentence is assigned some content.

The content of $A \wedge B$ is the set of analytic consequences *of* the union of c(A) and c(B), $c(A) \cup c(B)$, not that union itself.

The content of $A \vee B$ is $c(A) \cap c(B)$.

Entailment is now semantical defined as *containment of content*:

$v(A \rightarrow B) = 1$ iff $c(B) \subseteq c(A)$.

• To ensure a Relevant logic further postulates for content relations are laid down, e.g.

$c(A \vee B) = c(A) \cap c(B) \subseteq c(A)$.

This states that the content of the disjunction has to be in the content of each disjunct, i.e. cannot depend on just one disjunct. We have seen in the last Lecture that DJdQ is *prime*.

• An exception to this algebra is negation, since taking negation as *set complement* (like in standard Classical Semantics) would give

$A \wedge \neg A$

universal content, and would validate Explosion and *verum ex quodlibet* for entailment.

The negation of a sentence can be better understood as all those sentences that represent *alternative* situations, i.e. the content of ¬A are all facts the absence of which made A possible:

$$c(\neg A) \overset{\text{def}}{=} \{B \mid \neg B \in r(A)\}.$$

¬B∈r(A) means – given the definition of 'range' – that B has to be false as one precondition to establish A *analytically*.

This set is c(A)*, the complement set to a sentence A. One can prove c(c(A)*)* = c(A): the sentences making A false analytically being false analytically themselves establishes A being true analytically – therefore Axiom (9).

The interpretation *v* assigns c(A)* to ¬A. Negation, therefore, becomes an *intensional* junctor!

• The semantics of entailment – taking it as content containment – seems to be close to our natural understanding of entailment. Nothing about accessing the accessibility of worlds is involved.

This semantics – obviously – *presupposes* a theory of establishing the analytic consequences of a given sentence, i.e. a theory of analyticity! This may require meaning postulates, which in turn may contain expressions (like [→]) the semantics of which operates on contents! A meaning postulate – usually a conditional – may look like the prototypical entailment. *Basic* content assignment (at the beginning of the interpretation semantics) therefore may not be so basic at all.

• Notwithstanding this, Brady's logic seems to have some nice meta-logical properties:

DJdQ is sound and complete relative to the semantics of contents [cf. Lecture 5]. Sentences with identical contents can always be *substituted* for each other. [For contained meaning semi-substitutivity in a context holds.]

- As we have seen, several typical theorems do not hold in **DJ$^d$Q**:
  - Contraction [(A → (A → B)) → (A → B)]

    Even if the content of A → B is contained in the content of A – whatever that means – this *fact itself* does not analytically imply A → B. The stress is on 'analytically imply' as this the truth condition of an entailment [→]. Nested [→] are difficult to establish and understand in the entailment semantics. Often DJdQ may have (derived) rules on preservation of (logical) truth instead of theorems exhibiting nested entailments.
  - The Law of Excluded Middle (TND) [A ∨ ¬A]

    The content of A c(A) and c(A)* combined need not exhaust all possibilities as c(A)* is not the *set complement* of c(A), as seen above.
  - Disjunctive Syllogism [⊢¬A, ⊢A∨B ⇒ ⊢B] – as argued in previous Lectures.
  - The *Modus Ponens* Theorem [A ∧ (A → B) → B]

    The content of c(A) ∪ c(A → B) needs not *analytically imply* B, even though

    ⊢A, ⊢A→B ⇒ ⊢B – the *proof* of the preconditions allows logical closure
  - Negation Introduction (A → ¬A) → ¬A.

    Even if c(A) contains c(A)* – which hardly makes sense – this fact itself does contain c(A)*.

• In contrast, if we go through the list of axiom schemes they can be justified in terms of the semantics of entailment. [The rules need only be justified as preserving (logical) truth, which they clearly do.]

(A1) $A \rightarrow A$

Obviously, A can be analytically be established from A.

(A2) $A \wedge B \rightarrow A$

(A3) $A \wedge B \rightarrow B$

The union of contents c(A), c(B) contains c(A) and c(B)

(A4) $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$

If $c(B) \subseteq c(A)$ and $c(C) \subseteq c(A)$ it follows (analytically) that $c(B) \cup c(C) \subseteq c(A)$ by set theory.

(A5) $A \rightarrow A \vee B$

(A6) $B \rightarrow A \vee B$

The content of $A \vee B = c(A) \cap c(B)$ is a *subset* of the content of the disjuncts.

(A7) $(A \rightarrow B) \wedge (C \rightarrow B) \rightarrow (A \vee C \rightarrow B)$

If $c(B) \subseteq c(A)$ and $c(B) \subseteq c(C)$ it follows (analytically) that $c(B) \subseteq c(A) \cap c(C)$ by set theory.

(A8) $A \wedge (B \vee C) \rightarrow A \wedge B \vee A \wedge C$

Corresponds to set theoretic analytic statement: $c(A) \cup (c(B) \cap c(C)) \subseteq c(A) \cap c(B) \cup c(A) \cap c(C)$.

(A9) $\neg \neg A \rightarrow A$

By definition: $c(\neg\neg A) = \{B | \neg B \in r(\neg A)\}$ i.e. the set of all sentences which have to be false to analytically establish $\neg A$. c(A) is the set of sentences analytically implied by A. $c(A) \subseteq c(\neg\neg A)$, because A analytically implies that $\neg A$ is false. [Remember that Brady does not endorse explicit contradictions $A \wedge \neg A$.]

(A10) $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$

If the sentences that analytically exclude B are part of the content of A, then B being a fact analytically establishes that A cannot be the case.

(A11) $(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C)$

Corresponds to transitivity of subsets:

$c(C) \subseteq c(A)$ by set theoretical analytic implication from $c(B) \subseteq c(A)$ and $c(C) \subseteq c(B)$.

(A12) $(\forall x)P(x) \rightarrow P(á)$

The meaning of [$\forall$x] clearly implies the truth of an instance.

(A13) $(\forall x)(A \rightarrow P(x)) \rightarrow (A \rightarrow (\forall x)P(x))$

This and (A14), (A17), (A18) hold by the meaning of vacuous quantification: If for all objects the meaning of A analytically implies that P is true of that object where A does not say anything of that object ("x" is not free in A), then the meaning of A entails that P is true of all objects.

(A14) $(\forall x)(A \vee P(x)) \rightarrow (A \vee (\forall x)P(x))$

(A16) $P(á) \rightarrow (\exists x)P(x)$

By the meaning of [$\exists$x].

(A17) $(\forall x)(P(x) \rightarrow A) \rightarrow ((\exists x)P(x) \rightarrow A)$

(A18) $A \wedge (\exists x)P(x) \rightarrow (\exists x)(P(x) \wedge A)$