Lectures on Universal Logic

Lecture 5 – The Logic DJdQ. Outline & Proof Theory

Brady's Universal Logic is the logic DJdQ. "Q" stands for "quantificational". Quantification is less an issue between Relevant Logics and FOL (i.e. they basically agree on the semantics of the quantifiers) than the differences for propositional connectives. DJ is a Relevant Logic.

DJdQ has the connectives: $\neg$ [Non-Boolean Negation], $\vee$, $\wedge$, $\rightarrow$ [Entailment]

Other connectives can be defined by them, e.g. $A \leftrightarrow B \stackrel{\text{def}}{=} A \rightarrow B \wedge B \rightarrow A$

Starting from the usual definitions of formula, use of brackets etc. the logic can be stated as the following axiomatic system:

Axioms

1. $A \rightarrow A$
2. $A \wedge B \rightarrow A$
3. $A \wedge B \rightarrow B$
4. $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$
5. $A \rightarrow A \vee B$
6. $B \rightarrow A \vee B$
7. $(A \rightarrow B) \wedge (C \rightarrow B) \rightarrow (A \vee C \rightarrow B)$
8. $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$
9. $\neg\neg A \rightarrow A$
10. $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$
11. $(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C)$
12. $\forall xA \rightarrow A(y/x)$ where y is free for x in A
13. $\forall x(A \rightarrow B) \rightarrow (A \rightarrow \forall xB)$ where x is not free in A
14. $\forall x(A \vee B) \rightarrow A \vee \forall xB$ where x is not free in A
15. $A(y/x) \rightarrow \exists xA$ where y is free for x in A
16. $\forall x(A \rightarrow B) \rightarrow (\exists xA \rightarrow B)$ where x is not free in B
17. $A \wedge \exists xB \rightarrow \exists x(A \wedge B)$ where x is not free in A

Rules

1. $\vdash A, \vdash A \rightarrow B \Rightarrow \vdash B$
2. $\vdash A, \vdash B \Rightarrow \vdash A \wedge B$
3. $\vdash A \rightarrow B, \vdash C \rightarrow D \Rightarrow \vdash (B \rightarrow C) \rightarrow (A \rightarrow D)$
4. $\vdash A \Rightarrow \vdash \forall xA$

Meta-Rules

1. If $\vdash A \Rightarrow \vdash B$ (without using R4 on x free in A) then $\vdash C \vee A \Rightarrow \vdash C \vee B$
2. If $\vdash A \Rightarrow \vdash B$ (without using R4 on x free in A) then $\vdash \exists xA \Rightarrow \vdash \exists xB$

Rules (like *Modus Ponens* [R1]) state how to derive theorems from other theorems (including the axioms).

A Meta-Rule states how to derive theorems from other theorems given some previous derivation. The disjunctive character of the Meta-Rules accounts for the "d" in "DJdQ". Meta-Rules derive rules (not theorems directly): look at the consequent of MR1.

Initially $A \vee \neg A$ is not an axiom.

• The logic has way more axioms and rules than standard systems as the logic is a Relevant Logic based on relevant entailment, and not on [$\supset$].

[A semantic justification and foundation of each axiom and rule will be a topic in Lecture 6.]

• DJdQ can be presented also as a system of natural deduction. One main restriction consists in requiring that a premise to be discharged has been used. The system can use the usual rules of introducing and eliminating connectives [like ($\rightarrow$I), ($\rightarrow$E), ($\wedge$I), ($\wedge$E)] with the addition of a rule of DeMorgan-Distribution (see Axiom 8) and dropping the classical rule of ($\neg$I), which leads to Explosion (see below). ($\neg$E) is fashioned as: derive $\neg A$ from $\neg B$ and $A \rightarrow B$ (given the restrictions of premise use). The quantificational rules add the standard restrictions on generalization [for ($\forall$I)], and closely transpose into rules the Axioms 12 – 17. The Meta-Rules are transposed into modified Dilemma and ($\exists$E) rules requiring that the conditionals to be used are all part of the same proof (i.e. are not further nested in a sub-proof).

[The Universal Logic of Lecture 11 will be a natural deduction system and will make the whole approach more understandable, skipping the Relevance Condition though.]

• Many of the theorems of FOL can be derived in DJdQ (with [$\rightarrow$] replacing [$\supset$]).

These include the distribution of quantifiers, the typical theorems for [$\wedge$] and [$\vee$], as well as:

$$(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A)$$

$$\neg\neg A \leftrightarrow A$$

i.e. typical negation theorems like Contraposition and Double Negation Elimination, although TND does not hold.

Single substitution of equivalences in a context formula is also derivable (and thus permissible):

$$\vdash A \leftrightarrow B \Rightarrow \vdash ...A... \leftrightarrow ...B...$$

Addition of Classical Formula

For classical recapture (capturing all classical inferences in consistent contexts) classical formulae A', B' etc. can be added, for a system DJdcQ ("c" for "classical"). For these formulae one can add

Axiom 18 $\quad$ $A' \vee \neg A'$

Rule 5 $\quad$ $\vdash \neg A', \vdash A' \vee B' \Rightarrow \vdash B'$

resp. $\vdash A', \vdash A' \supset B' \Rightarrow \vdash B'$

A formula is classical iff it is bivalent, and TND holds. For classical formulae [¬] comes down to Boolean Negation.

Explosion then also is recaptured for classical formulae.

$$\vdash A', \vdash \neg A' \Rightarrow \vdash B$$

Thus, the theory of classes and sets involving the paradoxes must not use classical formulae.

Properties of DJdQ

DJdQ has (almost) the standard useful properties of a logic system. Some of these are established with respect to its non-standard possible worlds semantics [we look at in Lecture 6].

Important proof theoretic properties distinguish between truth preservation in 'regular worlds', in which all logical truths hold, and whatever might be in the irregular worlds. If we assume that the actual world is regular, we can be content with a meta-property holding just in the regular worlds.

(P1) (i) All (derived) rules of DJdQ are truth preserving at regular worlds.

(ii) All (derived) rules of DJdQ are truth preserving in light of the semantics of entailment.

(P2) All truth preserving rules can be derived in DJdQ.

The first property amounts to a form of correctness, the second to a form of strong deductive completeness.

(P3) The Deduction Theorem holds for DjdQ: if $\vdash A \Rightarrow \vdash B$ then $\vdash A \rightarrow B$

on the presumption that A (which may be a conjunction of premisses) has been used in the derivation (resp. all premisses have been used): no vacuous introduction of [→].

Avoiding vacuous discharge is a tenet of Relevant Logics to avoid forms of

$$\vdash A \Rightarrow \vdash (B \rightarrow A)$$

(P4) DJdQ is prime: $\vdash A \vee B$ requires $\vdash A$ or $\vdash B$

i.e. no TND-like (vacuously true) disjunctions are provable.

Although – as we have discussed in previous Lectures 2 and 4 – the Disjunctive Syllogism cannot simply hold, it can hold for theorems:

(P5) $\vdash \neg A, \vdash A \vee B \Rightarrow \vdash B$

DJdQ fulfils the Relevance Condition of variable sharing between antecedent and consequent of an entailment:

(P6) $\vdash A \rightarrow B$ only if A and B share a propositional variable (at some depth)

DJdQ also has some surprising and – in the light of our supposed understanding of entailment – controversial properties, like

(P7) $\nvdash \neg(A \rightarrow B)$

for any sentences A and B. One cannot prove any denial of an entailment statement!