Lectures On Universal Logic

Lecture 3 – Routley's Ultralogic In Action

Dialectical Set Theory

- One of the main applications of Routley's Ultralogic is Dialectical Set Theory (DST). Routley also applies his Ultralogic approach to the theories of probability and information, as well as to quantum theory, but of greatest philosophical interest is DST, which we deal with here.

[In the other fields the results are only/mostly 'negative' in the sense that some seemingly paradoxical results cannot be derived in an Ultralogic framework, whereas DST is claimed to be the better theory by its yield.]

- "A dialectical set theory is one which accepts the paradoxes of set theory as part of the theory" (911). This means that the supposedly deviant sets like the Russell Set or the Set of All Ordinals are sets within the theory, and the corresponding contradictions are at least in part derivable. Containing these proven contradictions (antinomies) the logic of DST has to be paraconsistent.

[In fact Routley does not exclude – at least in his early view – that DST might just be false, but at least worthwhile to investigate. The stronger position maintains that DST is true, and that one advantage of Ultralogic is its ability to deal with this theory without rendering it trivial, as Classical Logic does.]

- Some of the 'paradoxes' (like Russell's Paradox) result from the principle of Naive Comprehension (NC):

  For every property there is a set of those entities having this property.

In Second Order Logic (SOL), (NC2):

$$\forall F\ \exists y\ \forall x\ (F(x) \equiv x \in y)$$

In FOL as a schema (NC1):

$$\exists y\ \forall x\ (\varphi(x) \equiv x \in y)$$

NC looks intuitively plausible and captures, it seems, our idea of a set being the extension of a property. DST can take NC at face value, and has not to explain our intuitions away or provide a new picture of sets (like the iterative hierarchy of standard set theory ZFC).

• DST accepts that comprehension "is built into the meaning of *set*" (914). DST thus admits the Relevant Logic version of (NC1) as axiom (schema), DNC:

$$\exists y\ \forall x\ (\varphi(x) \leftrightarrow x \in y)$$

This does even allow "y" to occur in φ, which is usually excluded in NC.

• Taking φ to be [$x \notin x$] we get a set, call it "r", the Russell Set (of sets not being self-members), such that $x \notin x \equiv x \in r$. r being a set by universal instantiation (of ∀x) with r, we arrive at

$$r \notin r \equiv r \in r \qquad \text{or} \qquad r \notin r \leftrightarrow r \in r$$

'Russell's Paradox'. By a few steps of Classical Propositional Calculus (PC) we get from the lefthand version

$$r \notin r \wedge r \in r$$

an explicit contradiction (assume one side, Modus Ponens, Negation Introduction, do it again with the other side, then Conjunction Introduction).

Ultralogic contains principles for the relevant conditional [→] to derive the same contradiction from the righthand version:

1. $A \rightarrow A$
2. $A \wedge B \rightarrow A$ $\qquad A \wedge B \rightarrow B$
3. $A, A \rightarrow B \vdash B$
4. $A, B \vdash A \wedge B$
5. $A \vee \neg A$
6. $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$

The proof uses the definition of [↔] to simplify it with (2), starts – twice over, for both sides – with (1) [i.e. $r \in r \rightarrow r \in r$] proceeds with (6), applies (5) with (3), and sums up with (4).

These principles are plausible for a relevant conditional. Controversial can only be the addition of Tertium Non Datur (TND) – we come back to this in later lectures. TND is adopted by DST as another plausible principle on the extension of properties: either an object has a property or it doesn't: $\varphi(a) \vee \neg\varphi(a)$.

DST, therefore, contains contradictions.

• Given a contradiction $\perp$, Disjunctive Syllogism cannot be part of Ultralogic. By

$$\neg A \vee B, A \vdash B$$

any contradiction $A \wedge \neg A$ would immediately lead to Explosion and triviality. Thus Modus Ponens has to cast as above, involving [$\rightarrow$] not [$\supset$]. For the same reason Contraction

$$(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$$

cannot be adopted, leading to Explosion by way of Curry's Paradox [starting from DNC getting an instance $\forall x\ (x \in c \leftrightarrow (x \in x \rightarrow B))$, thus $c \in c \leftrightarrow (c \in c \rightarrow B)$, thus $c \in c \rightarrow (c \in c \rightarrow B)$, thus by Contraction $c \in c \rightarrow B$, thus $c \in c$, thus B – for any B]. Giving up Contraction makes Ultralogic even more restrictive than many relevant logics which 'only' give up rules like Disjunctive Syllogism.

• Ultralogic is the relevant first order system DKQ:

1. $A \rightarrow A$
2. $A \wedge B \rightarrow A$
3. $A \wedge B \rightarrow B$
4. $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee C$
5. $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$
6. $(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C)$
7. $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$
8. $\neg\neg A \rightarrow A$
9. $A \vee \neg A$
10. $\forall x\varphi(x) \rightarrow \varphi(a)$
11. $\forall x(A \rightarrow B) \rightarrow (A \rightarrow \forall xB)$ x not free in A
12. $\forall x(A \vee B) \rightarrow (A \vee \forall xB)$ x not free in A
13. $A, A \rightarrow B \vdash B$
14. $A, B \vdash A \wedge B$
15. $A \rightarrow B, C \rightarrow D \vdash (B \rightarrow C) \rightarrow (A \rightarrow D)$

16. $\vdash A \Rightarrow \vdash \forall x\, A$

[$\vee$] and [$\exists$] are defined as usual.

The principles seem clearly valid for an entailment conditional, and they are relevant.

Although DST contains contradictions, by the definition of [$\vee$] and (9): $\vdash \neg(\neg A \wedge A)$, i.e. DKQ contains a theorem usually taken to be the Law of Non-Contradiction!

DKQ has no extensional semantics. The semantics for [$\rightarrow$] is based on a ternary accessibility relation between (im)possible worlds. The semantics for [$\neg$] is based on the (in)famous 'Routley Star' (*), giving truth conditions of negations in term of 'witness worlds'. – Without going into details, we clearly have here a highly controversial semantics, quite in tension to Routley's appeals to intuitive principles in his central arguments for DNC and DKQ.

• DST is DKQ + DNC + Extensionality of Sets:

$$x = y \vdash x \in w \leftrightarrow y \in w$$

accompanied by a definition of identity for sets

$$x = y \overset{\text{def}}{=} \forall z\, (z \in x \leftrightarrow z \in y)$$

This cumbersome separate treatment of the two aspects of extensionality stems from problems of otherwise (i.e. by the simple: $\forall z\, (z \in x \leftrightarrow z \in y) \rightarrow x = y$) deriving formula, not-relevant in the terminological sense of relevant logic (like: $x = y \rightarrow (A \rightarrow A)$).

[If one is interested more in a dialectical set theory than in relevance one may work with NC/DNC and standard Extensionality of Sets instead.]

No further axioms are needed (see below).

• Given that in DNC the set being the extension of φ can be mentioned in φ, DNC allows for the condition $x \notin z$, for that very extension z of φ, so that we have:

$$x \in z \leftrightarrow x \notin z$$

for all x! This means:

- All objects belong to z iff they do not.
- All objects have at least one inconsistent property: Belonging-to-z.
- Thus, all objects are inconsistent!

The set z is obviously way more bizarre even than the set r.

• One advantage of DNC is that very strong set theoretical assumptions (like the existence of Choice Functions) need not be put as additional axioms (like an Axiom of Choice), but can be derived from DNC. All ZFC axioms can be gained from DNC.

Examples:

- The Pairing Axiom (of ZFC) is an instance of DNC: $\exists z\ \forall u\ (u \in z \leftrightarrow u = x \ \vee u = y)$
- The Existence of $\emptyset$ is an instance of DNC: $\exists z\ \forall u\ (u \in z \leftrightarrow \neg(u = u))$
- The Existence of Choice Functions (Axiom of Choice in ZFC) (in worlds where all logical truths hold): starting from DNC:

  $$\exists f\ \forall x\ (x \in f \leftrightarrow ((\exists u,v)(x = \langle u,v \rangle \wedge v \in u) \wedge \text{Function}(f))$$

  given the usual understanding of function and ordered pairs <u,v>, so for any set y: $\text{Function}(f) \wedge \exists z(z \in y) \rightarrow \langle y,f(y) \rangle \in f$ – a global choice function *f*

- The Powerset Axiom of ZFC, for any set x: $\exists z\ \forall u\ (u \in z \leftrightarrow u \subseteq x)$

etc.

• DST also allows to derive sets not available in standard set theories like ZFC.

One intuitive idea related to extensions is that a set should have a complement (the set of objects not belonging to it). Standard set theory does not allow this, as it does not allow a set of all sets, despite talking of ‘all sets’ and using universal quantifiers, as otherwise paradoxes like Russell’s are derivable. DST can deliver these sets.

Examples:

- The Universal Set V: $\exists z\ \forall u\ (u \in z \leftrightarrow u = u)$, giving rise to $V \in V$ and other ‘paradoxes’
- Given any set x the Absolute Complement exists: $\exists z\ \forall u\ (u \in z \leftrightarrow u \notin x)$

• DST is not constructive: all sets are ‘out there’, and not constructed by set theory and our definitions.

But in Noneism (Routley’s ontology of non-existing objects) this does not imply that they exist (somewhere)! Deriving the non-constructive choice theorems thus need not bother DST. Combining Ultralogic and Noneism makes DST a comprehensive powerful theory containing set theoretical reductions of all needed mathematical entities.

This is one more justification for the whole programme.

• We need a proof, however, that DST really is non-trivial. Just blocking some ways to Explosion and triviality does not guarantee that triviality might not ensue on another path. – We come back to this issue in a later lecture.

Manuel Bremer, 2020.