Recent Work on Frege

Michael Potter and Tom Ricketts. *The Cambridge Companion to Frege.* Cambridge: Cambridge University Press 2010. Pp. 639. (ISBN-13: 978-0-521-62479-4).

Richard G. Heck, Jr. *Frege's Theorem.* Oxford: Calrendon Press 2011. Pp. 308. (ISBN-13: 978-0-19-969564-5).

John Horty. *Frege on Definitions.* A Case Study of Semantic Content. Oxford: Oxford University Press 2009. Pp. 159. (ISBN-13: 978-0-19-973271-5).

In the last fifteen years a fast increasing interest in the history of early analytic philosophy has manifested itself. Many a book or collection covers the ground between about 1879 (the appearance of Frege's *Begriffsschrift*) and the 1930s. The Cambridge Companion series after some delay now involves a volume on Frege. This volume is not an introduction to Frege, but a 'companion' in the strict sense of supplementing and expanding one's prior understanding of Frege. An introduction to Frege had to contain a systematic exposition of Frege's theories, his formal systems, his important (logical) achievements and failures and might illustrate this with extensive quotes, as Frege is famous for his lucid style and argument. Surprisingly introductions to Frege in this style are hard to find. The *Cambridge Companion to Frege* certainly is no such introduction. It contains an introduction, but rather as a reminder on some central points of Frege's doctrines. Some of Frege's central theses (like those on sense and reference of an expression) are set out even for beginners in some of the articles, but many are not. What is strikingly absent in a volume on one of the founders of modern logic is a more comprehensive treatment of Frege's formal systems (that in the book *Begriffsschrift* [Bs] and that in the book *Grundgesetze der Arithmetik* [GgA]) and his logicist treatment of arithmetic. Despite the fact that the book contains a paper by Mark Wilson that outlines Frege's mathematical setting, which thus sits somewhat insulated in the collection. Most of the articles are non-formal and cover Frege's philosophical background to his philosophy of logic and mathematics. In this area they are mostly state of the art discussions and elucidations of central Fregean doctrines. Even readers familiar with Frege will find new perspectives and new or often overlooked approaches to classic Fregean topics. Many of the issues raised are controversial within Frege scholarship, but this only shows the fecundity of this type of historical exploration. An additional article by Peter Sullivan sets out Michael Dummett's interpretation of Frege, as Dummett's work gave rise to recent Neo-Fregeanism and set the tone of many debates on Frege in the last 30 years.

The first two chapters (by Joan Weiner and Warren Goldfarb) deal with Frege's concept of logic. The central issue is whether Frege has a meta-theory and meta-logic of logic or whether he follows the concept sometimes called 'logic as language', which takes logic not as a formal systems besides language, but as a partial regimentation of language for specific scientific (mostly mathematical) purposes. From our late 20th century knowledge of logic we often read our distinctions into the founders of modern logic, and this may be preposterous. For instance, modern logics often work with axiom schemata (i.e. the schema $A \supset (B \supset A)$ in fact talking about all sentences of this form), but Frege – at least in GgA – took the axioms as quantified sentences (i.e. $(\forall A,B)(A \supset (B \supset A))$ as Frege understood sentences as names of truth values and the system in GgA is a term logic, in which every term, including sentences, names an object). In the non-schematic view axioms are general truth about the *world*, whereas in the schematic view we talk about *sentences* being logically true. This is certainly philosophically important and has repercussions especially in the works of Russell and the early Wittgenstein. Nonetheless one should be cautious for two reasons. First, obviously the founders of modern logic had not its developed form at their disposal, especially model theory came up only somewhat later than Frege, although there are hints of model theoretic arguments already in Frege. Thus Frege could neither have our concept of logical truth in contrast to contingent truth, nor a corresponding formal semantic concept of consequence. Frege in fact argued at several points semantically taken a meta-logical standpoint (as noticed in the article by Richard Heck), and he proved the consistency of the system of Bs, a meta-logical result. Second, one has to distinguish the system of Bs and that of GgA: Bs in fact has axiom schemata as it does not have the substitution rule present in the system of GgA. The debate here offers some fascinating insights in the stepwise development of modern logic. The underdeveloped semantics explains Frege's stress on formal derivation and the stress on some (arithmetic) truths being analytic or of outmost generality (and thus logical truths).

Frege thought that his distinction between concepts (referred to by predicates) and objects (referred to by names) was one of his fundamental achievements. The papers by Alex Olliver and Thomas Ricketts explain its importance and its relation to Frege's account of sentential unity and the Context Principle (that only in the context of a sentence we can ask for the meaning of a word), which play a central role in Frege's justification of number expressions as proper names (in the *Grundlagen der Arithmetik* [GlA]). Famously Frege runs into *Kerry's Paradox*: Frege claims 'the concept *horse* is not a concept', because proper names (like "the concept horse") cannot refer to concepts! Frege's basic ontological distinction seems inexpressible, and Frege accordingly distinguishes between his formal systems, which as second-order systems clearly show the distinctions between the types of expressions, and elucidations (like 'no object is a concept') which introduce us to this distinction, but properly speaking make no sense – a distinction made famous by his follower Wittgenstein in the *Tractatus*. Frege turns here to talk about language to solve ontological riddles. This is one occasion of semantic ascent in Frege, and thus one of the inklings of analytic philosophy, others being the *Context Principle* and Frege's use of abstraction principles in analyzing the content of numeric statements.

Michael Kremer's and William Taschek's papers take on Frege's most famous thesis: the distinction between sense and reference. Kremer's paper is one of the outstanding contribution to the *Companion to Frege* as he manages not just to set out Frege's commonly known thesis for the umpteenth time without being boring, but rather relating its development to Frege solving puzzles and desiderata in his philosophy of logic and arithmetic. For instance, by introducing sense and given his theory of nominal definitions Frege can solve the problem of all logical truths because of being logically true implying each other, thus seemingly saying the same thing: axioms are those logical truths with the basic senses, retreating to which fosters our understanding of the derived theorems. Kremer claims in summary that we should consider the notion of sense as one of Frege's elucidations of the phenomena governing inferential relations. Sense appears as being more fine-grained than a model theoretic understanding of logical consequence. The concept of sense also plays a key role in Frege's epistemic strategy to explain numerical statements by explaining the sense of statements equivalent to them. Frege, however, is very inexplicit what sense ultimately is or how to understand it. Dummett read Frege's talk of 'mode of presentation' as the sense of a sentence being a verification rule. Many other accounts focus on the role of sense in indirect contexts (like 'believes that'). These theories and Frege's epistemological employment of sense invite one type of criticism of Fregean sense, namely having the work done by syntax (i.e. the form of the derivations or the syntactic features of the words themselves) which is done by Fregean sense. Frege's metaphor of 'grasping' a sense poses the problem of access to such abstract entities, which themselves are involved in our access to other entities. Taschek focuses additionally on the logical role sense has to play in Frege's framework and presents the traditional Russellian alternative of taking propositons/states of affairs as referential/informational content of a sentence and forsaking a semantic notion of sense. A topic later taking up again in Cora Diamond's paper on Wittgenstein and Frege, as Wittgenstein rejects both the idea of sense for subsentential units (sentences having sense as knowledge of their truth conditions) and of truth values as reference of sentences taken as names. Wittgenstein criticizes effectively the term logic of GgA, which has all functions (i.e. including sentential connectives) defined for all terms, which ultimately creates a tension with Frege's compositionality view on senses.

Papers by Michael Hallett (on Frege's relation to Hilbert), by Peter Hylton (on Frege's relation to Russell) place Frege's logicism in contrast to fellow logicist Russell and in contrast to formalist Hilbert. Frege corresponded for a time with both of them. Hallett covers both Frege's theory of definitions (rejecting contextual definition championed by Hilbert) and the resulting controversy with Hilbert on mathematical realism. This controversy leads us back to Frege's conception of logic and highlights Frege's use of quasi-model-theoretic permutation arguments. Hallett's paper covers some of the ground on Frege's account of definitions missing in Horty's book.

The only paper dealing squarely with Frege's logicism with respect to arithmetic (by Peter Milne) also unfortunately is the one where the author presents more of his own views than dealing with the debate on Frege. (One may also dislike a title like "Frege's Folly…" as misrepresenting the proportion of error and achievement in Frege, even in GgA.) Milne wants to trace the antinomy of GgA (the inconsistency derivable from *Basic Law V*) to Frege's insufficient theory of bearerless names. He presents a corresponding trilemma stemming from truth and negation commuting as prefixes even in sentences with bearerless names. He also claims, against Dummett and others, that "it is true that" does not yield an indirect context. This is dubious: it neither sits well with Frege's account of judgement (where a thought, i.e. indirect content, is the argument of a truth ascription) nor with Frege's ability to state that sentences containing bearerless names lack truth value (if saying of them that they are not true is not indirect, the whole sentence itself lacks a truth value and Frege's claim thus becomes inexpressible). The second part of the paper exposes the paradox stemming from *Basic Law V* and laudably stresses the conflict with *Cantor's Theorem*. Milne then sees some account of failure of reference (of a concept expression) as a way out of the paradox.

The *Companion to Frege* so invites and introduces readers who know some Fregean basics to a deeper understanding and exploration of Frege's philosophy of logic and language. Even Frege specialists may find surprising new turns of arguments. Readers looking for a treatment of Frege's logical systems have to turn elsewhere. A subsequent starting point for readers looking for Frege's logicism of arithmetic is Richard Heck's book *Frege's Theorem*.

Richard Heck's book *Frege's Theorem* collects several of his papers on Frege's logicism. Some of them have an added postscript in which Heck extends the original paper in light of the other papers in the collection. Heck's book covers some of the ground missing in the *Companion to Frege*. The book almost exclusively deals with Frege's logicism with respect to arithmetic. It presents important results of Frege and results about Frege's systems going beyond Frege with clarity and in technical rigor. The book thus presupposes a solid competence in logic and background knowledge of Frege.

The book *Frege's Theorem* focuses, obviously, on *Frege's Theorem*: that second-order logic supplemented with a cardinality principle also called "Hume's Principle" (HP) allows to derive the axioms of *Peano Arithmetic.* Frege achieved this in GgA, but the system of GgA is inconsistent. *Frege Arithmetic* is the core second-order logic of GgA minus the incriminated *Basic Law V*, which leads to *Russell's Paradox*, but supplemented with HP, which Frege derives with the help of *Basic Law V*. This system is consistent. (HP) states that the number of the A equals the number of the B if and only if there is a correspondence between the two concepts. If this principle is a principle of logic, then logicism has succeeded: *Frege's Theorem* asserts the reduction of arithmetic to logic. *Frege's Theorem* explains that the natural numbers form their structure by the involved notion of cardinality.

Richard Heck sets out the details around this theorem and explores its background and repercussions. He provides proofs of *Frege's Theorem* and variants in detail. He shows, for example, that weaker principles than *Basic Law V* suffice to derive (HP). Some of the issues Heck raises will fascinate specialists in the respective areas, but may convey to other readers the impression of the scholastic problem how many angels fit on a needle's pin. Some mayor issues are of interest to Frege scholars in general.

One mayor issue around *Frege's Theorem* is why Frege did not choose (HP) as an axiom, even after knowing of *Russell's Paradox.* Frege derives (HP) using his definition of number. He sees (HP) as insufficient to be an axiom as it does not solve the problem of transsortal identity. This problem has become (in)famous as the 'Julius Caesar problem'. The problem consists in an abstraction principle like (HP) being able to answer questions of the form '4 = ( )' for arguments which are not numbers, like '4 = Julius Caesar'. Obviously Julius Caesar is not the number 4, but our knowledge of this is not to the merit of (HP). Frege solves the problem by (i) having a definition of number, which does not apply to Caesar, because (ii) linked to the concept of value ranges (extensions) of concepts, which is a logical concept. Frege recurs thus only to our possession of logical knowledge and our apprehension of value ranges as logical objects to clarify general identity statements. Unfortunately the concept of value ranges is governed by *Basic Law V*, which is inconsistent given the rest of the system. If we take (HP) as an axiom in *Frege Arithmetic* the 'Julius Caesar problem' reappears. Frege finally forsook logicism as he could no longer reduce the concept of number to purely logical concepts (like correspondence function or extension). The problem sounds bizarre to the uninitiated, but for Neo-Fregeans or Neo-Logicists who do not want to exploit our empirical or non-logical knowledge that Caesar is not a number the problem has to be solved or to be dissolved (shown as irrelevant). Some stipulations that rule out Caesar as a number may do. The problem cannot be a formal one as *Frege Arithemtic* is provably consistent and *Frege's Theorem* holds. The problem ultimately points to the more important topic of the role of abstraction principle like *Basic Law V* and (HP). Frege uses them to introduce singular terms referring to individual abstract entities. They also are prime examples of his method of analysis: for instance, interpreting a number statement by interpreting an equivalent statement on correspondence (in case of HP), thus analyzing by interpretation, not decomposition of the original statement. All this again connects to Frege's *Context Principle*. (Two more systematic than historic chapters in Heck's book take on the broader ontological importance of such analysis in connection with the debate of reducing away abstract entities.)

A second mayor issue around *Frege's Theorem* is the question whether (HP) is analytic, i.e. whether – as Neo-Fregean claim – logicism can be vindicated after all. Heck argues for a limitation of (HP) to finite numbers being analytic. We have to be equipped not with the notion of a correspondence function but with the relation 'just as many'. One can derive a restricted form of *Frege's Theorem* then. Heck puts forth the – empirical? – claim that the original form of (HP) cannot underlie our arithmetical knowledge as no amount of reflection on the nature of arithmetical thought can assure us of its truth. The whole discussion, nonetheless, makes the traditional but questionable assumption that logicism is not vindicated if (HP) is synthetic. This thesis ultimately goes back to Frege (and others), who accept Kant's linkage between the synthetic and intuition. Given this, arithmetic, as Frege says in GlA, cannot be synthetic (*a priori*). What forbids, however, that our conceptual framework (the collection of reason's *a priori* principles) contains synthetic principles, where we understand synthetic as 'not true by definition'? One certainly needs a justification why to assume something as such a principle, but arguments why we posses such and such a concept of number we need anyway. The abhorrence of the synthetic *a priori* rests on a hostile appropriation of a Kantian legacy, which needs overhauling in several areas of analytic philosophy.

Heck's book though difficult, involving some repetition and at times byzantine on the spandrels of Fregean arithmetic institutes a standard on serious study of Frege's logicism. The importance of Heck's book lays in redirecting the attention of Frege scholars to the achievement of GgA. Frege's main work had a lot of bad press and was laid to the side because of the familiar story of *Russell's Paradox*. This image is misleading, as GgA contains not only a lot of Frege's philosophy of logic, but also contains in hidden form *Frege's Theorem*. A new English translation is in preparation, and one may hope for further advancement of Frege studies in this area.

The title of John Horty's book *Frege on Definitions* is misleading, as the book does not deal historically or systematically with Frege's theory of definition in general, but starts off from a specific *prima facie* puzzle about Frege's account of definitions and develops a theory of semantic content in the spirit of one understanding of Fregean 'Sinn'. Frege's theory of definition is one of nominal definitions, rejecting contextual definitions. Frege demands eliminability, non-creativity and determinacy (that the defined expression is defined for all arguments of the appropriate type) for a definition. Within second-order logic one can prove that eliminability and determinacy are equivalent (i) and (ii) that only nominal definitions fulfill determinacy and non-creativity. In one chapter of his book Heck shows that deciding the consistency of an arbitrary contextual definition is equivalent to deciding the satisfiability of an arbitrary second-order sentence, thus non-decidable. None of this is shown in Horty's book. Horty in fact starts from a part of Frege's theory which can be found in a posthumously discovered manuscript. Frege stresses here both the eliminability and the fruitfulness of definitions (in logic and mathematics). *Prima facie* this should puzzle us: How can something be both fruitful, but nonetheless be eliminable? Horty traces this tension through parts of Frege's writings on definition and sense. His central thesis is: Frege was following a representationalist theory of thinking (where thinking occurs either in inner speech [natural language representations] or in a *language of thought* [mental mostly sub-doxastic representations]). In a representationalist theory one realizes that human thinking is limited in processing speed, storage and working memory. Thus it can happen that two representations have the same content (Fregean sense), but differ crucially in complexity (i.e. their demand on our storage and processing capacities). What definitions achieve is clustering of content (in AI often called “chunking”). We are more able to think complex thoughts as some of the complexity is hidden in simple representations with complex content. For instance, we capture the complex definition of the integral in a symbol, and when we think using that symbol we do not have to represent the whole definition, but nevertheless know that we could expand the sense of the integral symbol if needed. Horty provides several passages in Frege which show that Frege work with such a representationalist conception. Horty then proceeds beyond Frege in developing a theory of sense (semantic content) which fits into this picture. Roughly following Dummett he understands sense in terms of procedures (of verification or proof or …). Senses are understood as ideal procedure to determine a referent. Processing a defined symbol then means processing a sub-routine (or being able to resort to a sub-routine). Horty outlines a corresponding categorical language and its composition rules. In this way syntactic features of the representation have psychological impact, although the representations share their meaning/sense.

Given this, however, one wonders again whether now we cannot *do away* with sense (say, given some Russellian or Wittgensteinian theory in which sentences do not refer to truth values but to states of affairs): Given some representation (be it in a natural, artificial or mental language) we have informational content and syntactic features, the difference of which explains a lot of what is explained by Fregean sense, as representationalist like Fodor have argued. Instead of saying that a representation has a procedure as sense one may better say that it is involved in procedures and/or serves as a label for a procedure. This way of talking doesn’t mention sense and semantics but stays on the procedural level. Several procedures may even determine the same result in different ways by involving different syntactic entities (representations). This syntax-centrism also solves a bewilderment which befalls Horty. At the very end of the book he wonders about the difference between semantics in the understanding of Fregean ‘sense’ (and thus reference to external, distal objects) and his procedural approach. This bewilderment simply rests on the abuse of ‘semantics’ in the theory of programming languages Horty unreflectively follows: The ‘semantics’ of an expression is understood there often (i.e. in case of most programming languages) as a procedure (carried out by the program involving that expression), and a procedure (whether now in assembler or machine language or whatever) is still a *syntactic* entity, Fregean semantics has dropped out of the picture. So, procedures as ‘semantics’ allow to found representationalism in material processes, which connect some device to its environment; a benefit adherence of a representationalist theory of mind stress. ‘Ways of presentation’ of a reference as Frege put ‘sense’ at least once, could be understood along these lines, but one should be clear then where one talks of representations themselves (including a crucial role of syntax) and where one talks about models (i.e. the semantics of the language in general). Horty’s book, although departing from its Fregean start off, thus provides an interesting idea for a representational theory. One can also see it as offering a general operational model for Dummett’s way of understanding ‘sense’.

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