Lectures on Paraconsistent Logics

[W]hen you have eliminated the impossible, whatever remains, *however* improbable, must be the truth.
As a rule the more bizarre a thing is the less mysterious it proves to be.

(Sherlock Holmes)

The hierarchy conception of semantics says we are always talking from some level in the hierarchy, and *at the same time* makes a general statement to the effect that constructing the semantics of a level we just go one level up. This is not just a contradiction. The situation is worse: If the theory is true something is impossible *to do*, but you just do it! Given the theory the adherents are *doing* something *impossible* in the strict sense.

(Talk at III. World Conference on Paraconsistency)

– so paraconsistency may be *the key*.

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Paraconsistent Logics

- These lectures give an overview on Paraconsistent Logics; at least they give a first overview including the main fields of paraconsistent logics, their basic characteristics, motivations and areas of application.
- Paraconsistent Logics are logics that allow for inconsistency. This "allowing for inconsistency" is spelled out formally. There are some variations within the field of paraconsistency how far this "allowing for inconsistency" should go.
- Allowing for that makes them definitely *non-classical*, even more so – one is inclined to say – than are the many valued logic approaches.
- Nonetheless such paraconsistent logics are worth considering:
  • Logical systems are worth considering *in their own right*, since so we learn about very abstract structural properties of logics and the concepts employed within them (e.g., negation, necessity, consistency ...)
  • Some non-classical logics are especially of interest from a philosophical perspective, since only by them some *philosophical* problems can be solved or even be stated.
- [The name "Paraconsistent Logic" is said to have originated with the Brazilian logician Miro Quesada (cf. Arruda 1989).]

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Paraconsistent Topics

- This introduction argues from a philosophical perspective why (some) paraconsistent logics are interesting or even the best candidates for treating some specific philosophical problems.
  This philosophical perspective is responsible for the focus laid on matters of semantic motivation and intuitive justification of some paraconsistent logic. Assessments and discussions try to meet philosophical scruples and misgivings hearing of paraconsistency.
- Although logic is seen from the point of view of its philosophical use, various formal systems are described, compared and employed.
- At the rate that the literature on paraconsistent logic and its various (supposed and real) applications is increasing it is, of course, not possible to deal with all approaches and all topics, but the topics dealt with here should at least give a first understanding of paraconsistent logics and their application, the major approaches and some trends in the discussion around paraconsistency.
- So – as always in introductions – knowing what is dealt with here may not be sufficient for understanding all of the literature, but for much of the literature it is necessary.

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Audience

- This introduction is written for:
  - Philosophers who have heard of these new systems, having some idea what they might be good for, but are still looking for a readable more general introduction.
  - Information scientists interested in the treatment of inconsistent data and their consequence relation.
  - Mathematicians who care about the foundational problems of set theory, or have even heard of "inconsistent arithmetic" or such things.
  - Linguists interested in formal methods and the philosophical foundations of semantics or the theory of formal languages.
  - Anyone interested in the paradoxes of semantics, set theory ..
- It is assumed that the reader has mastered a course in classical (or as I rather say "standard") First Order Logic, has heard at least in general about meta-logical properties of logical systems (like soundness) and about the idea of modal logics and their semantics (possible worlds...)
- It is surely helpful if the reader has some background knowledge in the philosophy of logic and language (especially semantics), or such philosophical fields as epistemology, ontology or meta-ethics.

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Content

- Apart from this first chapter which contains some definitions and some historical remarks on paraconsistency the lectures are divided into the following parts:

  - <u>Foundations</u>
  This part presents the basic philosophical motivation for paraconsistent logics (abbreviated "PL"), introduces some methodological guidelines for developing or choosing PLs and describes the different ways (i.e. systems) how one can develop PLs.

  - <u>Applications</u>
  This part employs some of the PLs or extensions thereof to deal with topics in set theory, meta-logic, semantics, epistemology, and meta-ethics.

  - <u>Discussions, Alternatives, Perspectives</u>
  This part confronts the whole approach with some problems (i.e., PL-ontology, hypercontradictions) and discusses some alternatives to PLs.

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Part I: Foundations

- 2. Semantic Closure
- 3. Methodological Considerations (& First Applications)
- 4. LP and Relatives
- 5. Relevant Logics
- 6. A Sequential Calculus
- 7. Adaptive Logics
- 8. LFI and Da Costa Systems
- 9. A Bunch of Other Systems

Part II: Applications

- 10. Semantics
- 11. Set Theory
- 12. Inconsistent Mathematics
- 13. Meta-Logic
- 14. Deontic Logic and Meta-Ethics
- 15. Epistemology and Philosophy of Science

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Part III: Discussions, Alternatives, Perspectives

- 16. Hypercontradictions
- 17. Inconsistent Ontology
- 18. Alternative Solutions of the Paradoxes?
- 19. Paraconsistency and Programming
- 20. Universal Logic

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Structure of a Chapter

- Each chapter contains a list of *further readings*; here the primary sources of the chapter are given and texts with which you might continue on the topic in question (general references are listed elsewhere)
- Almost all chapters
  - contain at the end one slide with some *questions* on the topic of that chapter inviting you to reflect on some of the issues or to grasp the crucial points more clearly
  - contain one slide with more formal *exercises* asking you to employ the methods just dealt with.
- The chapters in Part I are more or the less self-contained, the chapters in Part II obviously presuppose the PL-systems introduced in Part I.

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Notation and Usage

- I use mostly standard notation: $\forall\ \exists\ \neg\ \wedge\ \vee\ \cap\ \cup\ \supset\ \subset\ \subseteq\ \in\ \notin\ \neq\ \equiv$ ...
- "$\rightarrow$" is used for a conditional stronger than "$\supset$".
- "$\Rightarrow$" is used at the meta-level (e.g., in stating rules).
- "A", "B"... are meta-language expressions for object language formulas: "p", "q" ...
  [This distinction is under discussion here, so later expressions are translated into themselves at the meta-level.]
- Capital Greek letters mostly denote sets (of sentences).
  "Σ" is used for sums. "Δ" is used as truth operator.
- If there is some relation to Relevant logics I write "Relevantly" with a capital "R" (to distinguish from ordinary talk like "not relevant").
- By "logic", if not indicated otherwise, a proof system/theory *and* a corresponding semantics is meant.
  If the difference between syntax and semantics is important "proof system/theory" as usual means the systematization of the relation of derivability, $\vdash$, in contrast to the semantic consequence relation, $\models$.

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Notation and Usage (II)

- "$\vdash$" and "$\models$" are used with subscripts if a distinction between logics is needed in the context.
- By *proven equivalents* A and B it is meant: $\vdash A \equiv B$
  By *provably equivalent* in contrast it is meant that $A \vdash B$ and $B \vdash A$.
  Given *Modus Ponens* and the Deduction Theorem in some logical system, the proven equivalents are provably equivalent, but many paraconsistent logics lack either one or both of these!
  [This distinction becomes important for the validity of (respective versions of) replacement of equivalents in some logical system.]

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Notation and Usage (III)

- Proofs are often given in a Natural Deduction format without indenting or two dimensional rules [Chap. 7 & 20 provide more explicit details].
- **Example**: "< >" notes the assumptions a line depends on. We note the result of a (vertical) derivation in a (horizontal) formula by putting the assumptions mentioned in the dependency set of the last derived line on the left of "|—".

1.<1>	A	PREM
2.<2>	A ⊃ B	PREM
3.<1.2>	B	(⊃E) 1,2

that is A, A⊃B |—B.

The most simple case – by the way – shows the identity/idempotency property of "|—" and "|=" respectively:

1.<1>	A	PREM

yields by just stopping here A|—A, since the first line is at the same time the last line.

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Paraconsistency Is Contra-Intuitive?

- Consistency seems to belong to *the* most basic assumptions of logic and epistemology. How could anyone dare to put it into question? Can something that contains contradictions *make sense* at all?
- Even our ordinary concept of logical consequence is often based on the concept of consistency: A statement A follows from a set of statements Γ if $\Gamma \cup \{\neg A\}$ is inconsistent.
- But logical consequence can be defined without invoking consistency: A statement A follows from a set of statements Γ if in case that all $B \in \Gamma$ are true, A is true.
- Maybe one can introduce whole logical systems without relying on the concept of consistency (or consistency in a narrow sense).
- Furthermore, there are also powerful intuitions that speak against the treatment of inconsistency in standard logics, and powerful arguments that something might be wrong with the standard accounts how to avoid the inconsistencies that arise with the semantic paradoxes.

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Kinds of Inconsistency

- A set of statements is *simply inconsistent* iff it contains for some A both A and $\neg$A as elements; i.e. $(\exists A)(A \in \Gamma \wedge \neg A \in \Gamma)$.
- A set of statements is *syntactically absolute inconsistent* (or *trivial*) iff it contains all A (or a given language) whatsoever, i.e. $(\forall A)(A \in \Gamma)$. We also say that a set of statements is *syntactically absolute inconsistent* (or *trivial*) iff its set of consequences (given some specified logic) is *syntactically absolute inconsistent* (or *trivial*); i.e., $(\forall A)\ \Gamma \vdash A$
- A set of statements is *semantically absolute inconsistent* (or *trivial*) iff it makes all statements A true. We also say that a set of statements is *semantically absolute inconsistent* (or *trivial*) iff its set of consequences (given some specified interpretation) is *semantically absolute inconsistent* (or *trivial*); i.e. $(\forall A)\ \Gamma \models A$.
- A set of statements is *explicitly inconsistent* iff $(A \wedge \neg A) \in \Gamma$.
- A set Γ may be simply inconsistent without being explicitly so, and be explicitly inconsistent without being absolutely inconsistent.

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Paraconsistency

- A logic is *paraconsistent* iff a set of statements can be simply or explicitly inconsistent without being trivial with respect to the concepts of $\vdash$ and $\models$ defined by that logic.
- The decisive property is: $(\exists A)(A \in \Gamma \wedge \neg A \in \Gamma) \wedge (\exists B)\ \Gamma \nvdash B$
- Triviality is certainly bad: If everything is true all distinctions disappear, nothing can be asserted in distinction to anything else. Triviality has to be avoided. This requirement does not carry over to inconsistency if inconsistency and triviality can be kept apart.
- Logics that cannot distinguish between inconsistency and triviality are called *explosive*, since a simple inconsistency causes a set (of consequences) to explode by containing all sentences of the language. Paraconsistent Logics (PLs) can, therefore, also be characterized as being *non-explosive*.

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ex contradictione qoudlibet

- Standard logics (i.e. FOL, but also Intuitionistic and Minimal Logic, systems based on them, and many Many Valued Logics) are explosive by principles/rules that cause explosion. The most (in-)famous is *ex contradictione qoudlibet* (sometimes called "principle of explosion"):

  $A, \neg A \vdash B$ or

  $(A \wedge \neg A) \vdash B$

  i.e. given a (simple) inconsistency any formula B whatsoever can be derived. The proof in PC:

1.<1>	p	PREM
2.<2>	$\neg p$	PREM
3.<1>	$p \vee q$	$\vee$I, 1
4.<1,2>	q	$\vee$E, 2, 3

- Avoiding explosion means giving up as theorems formula like:

  (1) $p \wedge \neg p \supset q$ and

  (2) $p \supset (\neg p \supset q)$

  [both proven in variants of the proof above with ($\wedge$E) or ($\supset$I) used.]

ex contradictione qoudlibet (II)

- Avoiding such theorems and rules like *ex contradictione qoudlibet* requires changing the logic in such a way that they are no longer derivable.
- Given the proof above this means dropping one of the rules employed, i.e. ($\vee$I) or ($\vee$E), or inventing some new restrictions on a derivation being a *proper* derivation (like the restrictions used in the context of ($\forall$I) and ($\exists$E)).
- Many paraconsistent logics give up ($\vee$E), the so called *Disjunctive Syllogism*. Some lay down further requirements on proofs (typically such that the consequence should be *Relevantly* related to the premises). Some combine these procedures.
- In any case such revisions should be justified by
  - their effectiveness in blocking explosion
  - their power in blocking further contra-intuitive theorems
  - showing that not too much is being given up (the simplest PL is such that *nothing* can be derived at all from some Γ!)

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Weak Paraconsistency

- The *weak* paraconsistent approach says that we need a paraconsistent logic to deal with inconsistencies, especially with inconsistent theories, although weak paraconsistency sees inconsistencies still as something bad that should be *avoided* and has to be avoided in the end.
- The main motivation for weak paraconsistency lays in the fact that many of our theories (or systems of belief) are simply inconsistent. Nevertheless we do not conclude everything from them.
- Sometimes we discover that our system of beliefs *was* inconsistent (say by believing that Schuhmacher did not win at Monza, that a Ferrari driver won, that Barricello did not win and knowing in general that every Formula I team has two pilots only); we correct ourselves, but we do not remember that at the time of inconsistency we believed every statement whatsoever (e.g. that John Paul II won at Monza). Probably our current belief system *is* inconsistent, but we do not feel inclined to believe everything. Applied to our ordinary reasoning, therefore, explosion is a *highly contra-intuitive* logical principle.

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Weak Paraconsistency (II)

- The argument concerning belief systems may be circumwented if one claims that ordinary belief is not deductively closed. That is, at least, controversial, but an ideal reasoner should aspire to closure.
- In case of scientific theories, however, the theory *as a structure* can be considered. And here theories that are inconsistent are not always considered to be trivial. Weak paraconsistency claims that this often was and often is the case in all the sciences.
- An often cited example is Bohr‘s theory of the atom: It stated that an electron can revolve around the kernel without giving off energy, and at the same time it made use of Maxwell‘s equations, which imply that an accelerated electron (like the one revolving around the kernel) has to give off energy. The theory, so, is inconsistent. Nevertheless nobody claimed arbitrary things like the electron receiving energy and so on.
- Weak paraconsistency tries to model these cases.
- Weak paraconsistency does *not* claim that the logic of completed science has to be a paraconsistent one.

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Strong Paraconsistency

- The strong paraconsistent approach says that contradictions are not things that happen, but can be avoided. According to strong paraconsistency (some) contradictions are unavoidable. Some contradictions can even be proven. Some contradictions, therefore, have to be considered as being *true*!
- *Dialetheism* is the claim that there are true contradictions (called "dialetheias" by Routley and Priest).
- Of course, dialetheism is the philosophically more controversial and challenging position. Its justification needs even closer scrutiny than that of weak paraconsistency. Its main motivations are the idea of semantic closure and universality (including mostly the adherence to naive set theory).[see below]

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The Law of Non-Contradiction

- The strong paraconsistent approach – and the weak paraconsistent approach at least in some sense – seem to reject the Law of Non-Contradiction (LNC). – But what exactly does this "law" state?
- The LNC can be taken as a *syntactic* requirment to derive as a theorem $\neg (A \wedge \neg A)$, it may be taken as a *semantic* requirement on evaluations of formula, it may be taken as a *metaphysical* principle – or it may be taken as a principle of discourse felicity.
- Being unclear what the LNC is strong paraconsistency can be stated – and understood – as the claim that for some sentences both the sentence itself and its negation are *true* at the same time.
- This claim is *compatible* with the formula $\neg(A \wedge \neg A)$ being derivable, as we will see soon [see Chap. 4]. LNC may be syntactically in force.
- This claim is *compatible* with ordinary objects having or not having their properties [see Chap. 17]. LNC covers *some* metaphysical truths.
- This claim is a *specific* assertion, as is the claim that for some sentences both the sentence and its negation are derivable, since this *excludes* that the sentence is not derivable in the logic under review.

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The Law of Non-Contradiction (II)

- An adherent of (some version of) LNC can force us to accept (a version of) LNC *by defining* a contradiction as a sentence that cannot be true.
- An example may be a *bottom particle* $\perp$ (defined to have always only the truth value "false"). Paraconsistent Logics allow introduction of $\perp$.
- This, however, is only a verbal manoeuvre. The dialetheist may agree that contradictions *understood thus* are not true, but that there is an interesting class of sentences (like the Liar) which are true and false at the same time. (The proposed definition being revisionary, indeed.)
- There may be a strong (binary) negation present in our language; the dialetheist "just" claims that this does not guarantee attributing truth or falsity *exclusively* to *all* sentences.
- Dialetheism can (always) be rephrased as the claim *that there are sentences that are true and false at the same time*, so that – given some intuitive understanding of negation – they and their negations are both true. For these concepts of truth and negation – we obviously have – an equivalent of the just introduced LNC does not hold.

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Strong vs. Weak Paraconsistency

- Note that weak and strong paraconsistency can agree widely on the logics under investigation, since both try to avoid explosion. Their difference concerns the (ultimate) *truth* of contradictions.
- If dialetheism claims that some contradictions can be proven and weak paraconsistency denies the (ultimate) truth of contradictions that can only mean that weak paraconsistency has to reject the systems and logics that allow these proofs.
- That might mean that weak paraconsistency has the shortcomings of *both* worlds: Having neither the strength of FOL nor semantic universality (or other benefits of dialethism).
- Dialetheism, on the other hand, has to do some serious explanation *how* it can be that there are true contradictions and whether this implies that there are inconsistent *objects* (apart from sentences which are true and false at the same time)!
- This introduction sides with dialetheism and tries to impart some under- standing of its motivations, working and use.

The Prehistory of Paraconsistency

- Philosophers and scientists contradict each other and themselves quite often, but that does not make them dialethist or adherents of PL.
- Some philosophers have countenanced views sympathic to the occurence of true contradictions, for example Cusanus with respect to the properties of God. Hegel is famous for the ubiquity of contradictions in his writings, contradictions in human thought or world history. Meinong proposes an ontology of inconsistent objects.
- The mere support of true contradictions does not entail that the corresponding philosophers think there is a logic of contradictions.
- Hegel might be considered an exception in as much as his dialectic in his main work *Die Wissenschaft der Logik* [The Science of Logic] is supposed to proceed by developing and cancelling contradictions. Hegel, however, didn‘t know much logic. The dialectic proceeds as a "logic" of concepts. Even under adherents of Hegel it is still unclear how it really proceeds, and whether there is a general method present that could in principle be formalized.

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The History of Paraconsistency

- The history of paraconsistency begins with those who really develop logical systems that deal with the occurence of contradictions.
- The earliest work are some articles by Vasilev (about 1910) where he proposes a syllogism allowing for S to be P and non-P.
- Jaskowski in 1948 published a first paper on the logic of discussions ("Discursive Logic"). In a discussion people may contradict each other. Someone taking down notes cannot simply build the conjunction of everything said, since this would often result in $A \wedge \neg A$. Jaskowski develops a logic that takes as given what each speaker says without ending in triviality [see Chap. 3].
- DaCosta in the 1960s develop a new type of inconsistency tolerant logics. Ansenjo did related work in 1954. [Chap. 8 deals with them.]
- The first system of Relevant Logic was given by Orlov in 1929. Relevant Logic as we know it was developed in the USA and in Australia by Anderson, Belnap, Dunn, Routley, Meyer et. al. in the 1960s. [see Chap. 5, but many systems here are Relevant Logics].

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The Present of Paraconsistency

- Today paraconsistency is a rapidly developing field of non-standard logic. Three World Conferences on Paraconsistency, several readers and a plentitude of papers are witness to that.
- There are three main camps of paraconsistent logic and a lot of work not within exactly one of these:
  - the Brazilian approach related to the work of Newton DaCosta and championed by Walter Carnielli and his co-workers [see Chap. 8];
  - the Adaptive approach championed in Ghent by Diderik Batens and his team [see Chap.7];
  - the Relevant approach and his relatives championed by the co-authors of the late Richard Routley (Sylvan): Ross Brady, Robert Meyer, and especially Graham Priest, who for the wider philosophical audience has become *the* author/speaker of the field [see Chap. 4 – 6].
- Within an introductory text it is difficult to do justice to all recent developments and ideas, but it may be hoped that some are covered and the rest at least require covering the ground presented here.

Questions and Exercises

- This was just the introduction, so there was not much to ask you about.

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Further Reading

- The whole presentation draws heavily in almost *each* chapter on the following resources:
- Manuel Bremer. *Wahre Widersprüche*. Eine Einführung in die parakonsistente Logik. Sankt Augustin, 1998.
- Graham Priest's books and papers; I especiall recommend his *In Contradiction*. A Study of the Transconsistent. Dordrecht, 1987.
- Priest, Graham/Routley, Richard. *On Paraconsistency*. Research Report 13. Australian National University 1983; a newer concise overview from Priest's perspective is: Priest, Graham. "Paraconsistent Logic", in: Gabbay, D./Guenthner, F. (Eds.) *Handbook of Philosophical Logic*. 2nd Edition., Vol.
- On the history of paraconsistency also consult these resources; on Hegel see (Bremer 1998, Chap. 6.1).
- For some background on the different fields of philosophical logic and formal semantics we touch upon on the way an excellent help are the essays in: Goble, Lou (Ed.) *The Blackwell Guide to Philosophical Logic*. Oxford, 2001.

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