Contradictions in Mathematics

Contradictions are typically seen as anathema to mathematics. As formalism sees consistency as the only condition to consider some mathematical structure as an object of study, inconsistency becomes the single exclusory criterion. For mathematical realists accepting inconsistencies comes down to accepting inconsistent objects, which just seems bizarre.
In this paper I consider two inconsistency friendly approaches in (the philosophy of) mathematics. In a recent study *How Mathematicians Think* William Byers argues that one way of mathematical progress is by way of contradiction. The first paragraph outlines Byers' thesis, but it turns out that contradictions play a role only *ex negative*. In contrast to that the approach of inconsistent mathematics claims contradictions to be real. Especially in inconsistent arithmetic contradictions are said to play a vital role. They turn out to provide a framework for a finitist position which endorses inconsistent numbers.

§1 Byers' Creative Use of Contradictions

In *How Mathematicians Think* (2007) William Byers claims mathematics to at core a creative activity. Mathematical reasoning, according to Byers, is not primarily algorithmic or based on proof systems, but on using (great) 'ideas' to shed new light on mathematical objects and structures. These ideas not only are placed at the centre of mathematical understanding, which Bryer calls 'turning on the light', but also propel mathematical progress. Bryers presents a couple of examples in which a crucial step forward in the development of mathematics depended on the presence of two at first sight unrelated or even barely compatible perspectives on some mathematical structure. He starts with the discovery of the irrational numbers (like $\sqrt{2}$), where $\sqrt{2}$ is clearly present as a geometric object (the length of the hypothenuse of the right angled triangle with unit length sides) but is not allowed for by (early Greek) arithmetic. The real numbers 'provide a context' (38) in which the two perspectives are unified. The core of mathematics, according to Bryers, is finding such situations and being able to understand them by providing a more comprehensive view. This process is creative and not algorithmic.
One of the central methodological concepts – besides ambiguity – Byers uses in analysing the examples he presents is 'contradiction'. The very subtitle of his book reads 'Using Ambiguity, Contradiction, and Paradox to Create Mathematics'.
'Contradiction' is understood by Byers in two ways. On the one hand we have two seemingly contradictory perspectives in some of the mathematical problems he presents. For example, one may see $\sqrt{2}$ as a decimal, 'an "infinite" indefinite object' (97), but also as a finite geometric object. One can see '$2 + 3 = 5$' both as expressing a fact of identity (i.e. something static) as well as expressing the process of adding (i.e. something dynamic). But of course the fact can be established by going through the process of adding, the two perspectives are finally compatible and not inconsistent. The paradox of zero (as something that is nothing) vanishes with axiomatization.
On the other hand some seminal proofs work by using contradictions, or so Byers claims. Famous examples are Cantor's use of diagonalization or Gödel's Theorems.
Now, if we look at these examples it becomes obvious that none of the mathematicians in question endorses any of the contradictions. Quite the opposite. In these indirect proofs contradictions are used as a threat to establish the opposite result. Just for reduction some innocent looking assumption is made which turns out to be contradictory and thus untenable. What we really see here is not a creative use of contradictions, but the creative use of indirect proof methods. Mathematics still avoids the contradictory. Even supposedly incompatible

perspectives on one and the same structure have to be kept distinct from contradictions. The paradoxical calls for resolution.

§2 Inconsistent Mathematics and Finitism

To have *an inconsistent number theory* means at least that within the theorems of number theory there is some sentence A with A being a theorem and ¬A being a theorem at the same time. Supposedly this contradiction corresponds to some object/number *o* being an inconsistent object. So inconsistent mathematics is connected to inconsistent ontology. Its underlying logic has to be paraconsistent (cf. Bremer 2005).

Given Compactness of **FOL** one can prove that there are non-standard models of arithmetic, which contain additional numbers over and above the natural numbers. These additional numbers behave consistently, however. Consistency provides them in the first place.

Inconsistent arithmetic may concern itself with the opposite deviance: Having arithmetics where there are less numbers than in standard arithmetic.

This is of outmost philosophical interest, since the infinite is a really problematic concept leading to the ever larger cardinalities of "Cantor's paradise", and finitism (in the sense of the assumption that there are only *finitely many* objects, even of mathematics) is therefore an option worth exploring and pursuing.

Inconsistent Arithmetic that are finite may have any finite size you like. They contain one largest number. Since we do not know which number really is the largest we may assume that one of these arithmetics is true, although we don't know which. Let n be some natural number, let N_n be a set of arithmetic sentences. These have the following properties (cf. Priest 1994):

(i) $N \subset N_n$.

(ii) N_n is inconsistent.

(iii) $A \in N_n$ for a (negated) equation A concerning numbers < n if and only if $A \in N$.

(iv) N_n is decidable.

(v) N_n is representable in N_n (thus we have a N_n truth predicate).

(vi) For the proof predicate B( ) of N_n every instance of B("A") ⊃ A is in N_n.

(vii) If A is not a theorem of N_n ¬B("A")$\in N_n$.

An inconsistent arithmetic N_n thus has quite remarkable properties:

- by (i) we have that N_n is complete, since N is.
- by (iv) it has all the nice properties that N does not have, although N_n is complete!
- by (v) we can in the language of arithmetic define a truth predicate.
- by (vi) N_n has an ordinary proof predicate.
- by (vii) in conjunction with (iii) we have not only that N_n is not trivial (by excluding some the equations that are excluded by N), but that this non-triviality can be established within N_n itself.

A model of a theory N_n is constructed as a filtering of an ordinary arithmetic model. The trick in case of N_n is to chose the filtering which puts every number < n into its equivalence class, and nothing else; and puts all numbers ≥ n into n's equivalence class. As a result of this for x < n the standard equations are true (of [x]), while in case of y ≥ n *everything* that could be said of such a y is true of [n]. So we have immediately n = n (by identity) and n = n + 1 (since for y = n + 1 in N this is true). The domain of a theory N_n so is of cardinality n. n now is an inconsistent object of N_n. If for the moment we picture the successor function by arrows we can picture the structure of a model of N_n thus:

$$0 \rightarrow 1 \rightarrow \dots \rightarrow \overset{\curvearrowleft}{n}$$

Such models are called "heap models".
The logic modelling N_n has to be paraconsistent. And it has to have restrictions on standard first oder reasoning as well.
The *Löwenheim/Skolem-Theorem* is one of the limitative or negative meta-theorems of standard arithmetic and **FOL**. It says that any theory presented in **FOL** has a *denumerable* model. This is strange, since there are first order representations not only of real number theory (the real numbers being presented there as uncountable), but of set theory itself. Thus the denumerable models are deviant models (usually Herbrand models of self-representation), but they cannot be excluded. Given the general procedure to finitize an existing mathematical first order theory using paraconsistent semantics, there is a paraconsistent strengthened version of the *Löwenheim/Skolem-Theorem*:

> Any mathematical theory presented in first order logic
> has a *finite* paraconsistent model.

§3 The Benefits of Inconsistency

A mathematics that does not commit us to the infinite is a nice thing for anyone with reductionist and/or realist leanings. As far as we know the universe is finite, and if space-time is (quantum) discrete there isn't even an infinity of space-time points. The largest number may be indefinitely large.
If there are inconsistent versions of more elaborated mathematical fields like the calculus one may draw some general philosophical conclusions:

- If there are corresponding inconsistent versions of these mathematical theories with comparable strength to the original theories then consistency is not the fundamental mathematical concept, but functionality (of the respective basic concepts) may well be.
- If the justification of mathematics depends on its applicability and the inconsistent versions are of comparable applicability then they are justified not just as mathematical theories, but even in the wider perspective of grasping fundamental structures of reality; there no longer will be available the argument from mathematical describability to the consistency of the world.