An ε-Calculus

The ε-calculus was part of Hilbert's finitist programme, but can be used, of course, independently of its intended application.
[Hilbert's, Bernay's and Ackermann's use of it will be of no concern here.]

The ε-operator or ε-symbol is syntactically like a ι-operator or ι-symbol.
Whereas the ι-operator is used to define a definite description to which one object corresponds (in standard FOL with descriptions) or at most one existing object (in Free Logic description theories) the ε-operator picks one out of the set of objects satisfying the predicate in question.

Like standard description theories employ a dummy object for those descriptions which do not pick out a unique object, the semantics of an ε-calculus will employ a dummy object in case the ε-operator is applied to a predicate which denotes the empty set.
The ε-expression thus involves choosing 'one of the …'. Which object is chosen stays indefinite, and we do not have to know how many 'of the …' exist. The choice is not by us, but considered to be one of the possible choices among the objects satisfying the predicate in question. One may also think of it as a choice left open to an opponent in an argument.
In the formal semantics of an ε-calculus an ε-expression like 'εxF(x)' denotes either one of the objects *d* of the domain *D* such that $d \in ||F||$ or some arbitrary but fixed dummy object $d_0 \in D$ in case $||F|| = \emptyset$. This means that the expression 'ε' itself is interpreted by a choice function on the subsets of the domain D, d_0 being the fixed choice for $\emptyset$.

Semantically an ε-calculus thus employs a *global choice function*. Models for the ε-calculus extend the usual FOL-models with a global choice function f (one for each model, of course). The truth conditional semantics contains the further denotation condition (with φ being schematic for a predicate): $||\varepsilon x \varphi(x)|| = f(||\varphi||)$.

A calculus like the following using ε-expressions is a *conservative extension* of FOL (i.e. with respect to 'ε'-free formula FOL and the ε-calculus coincide). [Hilbert/Bernay's *Second ε-Theorem*]

Axioms and rules of an ε-calculus

[following Leisenring's axiomatization, aiming at easy use, not at compression]
Usual syntax with: A, B … wffs; s, t … singular terms; F, G … predicates/open formulas.

Axioms

A1 $A \supset (B \supset A)$
A2 $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
A3 $(\neg A \supset \neg B) \supset (B \supset A)$
A4 $(A \supset \bot) \supset \neg A$
A5 $(A \wedge B) \supset A$
A6 $(A \wedge B) \supset B$
A7 $A \supset (B \supset A \wedge B)$
A8 $\neg(A \vee B) \supset \neg A$
A9 $\neg(A \vee B) \supset \neg B$
A10 $\neg A \supset (\neg B \supset \neg(A \vee B))$

A11 $\forall x F(x) \supset \neg\exists x \neg F(x)$
A12 $\neg\forall x F(x) \supset \exists x \neg F(x)$
A13 $\neg\exists x F(x) \supset \neg F(t)$
A14 $\exists x F(x) \supset F(\varepsilon x F(x))$

A15 $(s = t \wedge F(s)) \supset F(t)$
A16 $t = t$

A17 $\forall x(F(x) \equiv G(x)) \supset \varepsilon x F(x) = \varepsilon x G(x)$

Rules

R1 Uniform Substitution (S)
R2 $\Gamma \vdash A, \Sigma \vdash A \supset B \rightarrow \Gamma \cup \Sigma \vdash B$ *Modus Ponens* (MP)

Once we have ε-expressions there is only a use for ι-expressions in cases where uniqueness is of importance. We can simply add ι-expressions to an ε-calculus as they can be defined by an ε-expression. Using ι-expressions standard FOL description theories use a dummy object for failing descriptions. Define "$\exists!xF(x)$" by "$\exists x(F(x) \wedge \forall y(F(y) \supset y=x))$". Let us again use the object d_0 denoted by the expression "d". We can add the following definition to the calculus:

(Dι) $\iota x F(x) \stackrel{\text{def}}{=} \varepsilon x((F(x) \wedge \exists!xF(x)) \vee (\neg\exists!xF(x) \wedge x=d))$

Comments:

1. (A1) – (A10) deal with propositional logic.

2. (A11) – (A14) deal with quantificational logic, and we see here an employment of an ε-expression in specialising an existence assumption. In principle employing 'ε' allows for the reduction of the quantifiers [Hilbert/Bernay's *First ε-Theorem*], by the two definitions:

$$\exists xF(x) \equiv F(\varepsilon xF(x))$$
$$\forall xF(x) \equiv F(\varepsilon x\neg F(x))$$

and using the axiom: $F(t) \supset F(\varepsilon xF(x))$. The calculus here, again for ease of use, employs both quantifiers and 'ε'.
By (A12) and (A14) we get

$$\neg\forall xF(x) \supset \neg F(\varepsilon x\neg F(x))$$

and thus by contraposition, (A3) and MP, we get

(T1)	$F(\varepsilon x\neg F(x)) \supset \forall xF(x)$	
(DR1)	$\Sigma\vdash F(\varepsilon x\neg F(x)) \rightarrow \Sigma\vdash\forall xF(x)$	(UG)

Semantically speaking $F(\varepsilon x\neg F(x))$ means that even 'one of the $\neg F$' is F, which, avoiding the obvious contradiction $F(\varepsilon x\neg F(x))\wedge\neg F(\varepsilon x\neg F(x))$, can only mean that $\|\neg F\|=\varnothing$. So $\|F\|=D$ and $\varepsilon x\neg F(x)=d_0$. Note that, correctly, $d_0\in\|F\|$ as $\|\neg F\|=\varnothing$.
By (A11) and (A13) we get

(T2)	$\forall xF(x) \supset F(t)$	
(DR2)	$\Sigma\vdash\forall xF(x) \rightarrow \Sigma\vdash F(t)$	(US)

By (A13) and contraposition weg et

(T3)	$F(t) \supset \exists xF(x)$	
(DR3)	$\Sigma\vdash F(t) \rightarrow \Sigma\vdash\exists xF(x)$	(EG)

We can also put (A14) in form of a derived rule

(DR4)	$\Sigma\vdash\exists xF(x) \rightarrow \Sigma\vdash F(\varepsilon xF(x))$	(ES)

3. (A15) and (A16) are the usual axioms extending FOL with identity. (A15) can also be expressed as derived rule:

(DR5)	$\Sigma\vdash s=t, \Gamma\vdash F(s) \rightarrow \Sigma\cup\Gamma\vdash F(t)$	(SI)

4. (A17) corresponds to the global choice function being a function, i.e. being unique in choice of a representative of a subset of D, irrespectively which predicates denote this subset.

5. Using (Dι), (ES) one can prove the usual theorems for definite descriptions:

(T4)	$\exists!xF(x) \supset F(\iota xF(x))$
(T5)	$\neg\exists!xF(x) \supset \iota xF(x) = d$

The calculus in question is correct and deductively complete w.r.t. the usual FOL semantics extended with the semantics for ε-expressions. The Deduction Theorem holds as well.